# ECE 486           Second Midterm Exam: Sample

This is a *closed-book closed-notes* test, no calculators allowed. Some useful facts are summarized in the back of the exam. For scratch work you may use backs of pages and the additional sheets provided. You have five problems to solve in 80 minutes.

Good luck!

**Full Name:** ...............................................................................

Problem 1:   .....   / 15 points

Problem 2:   .....   / 15 points

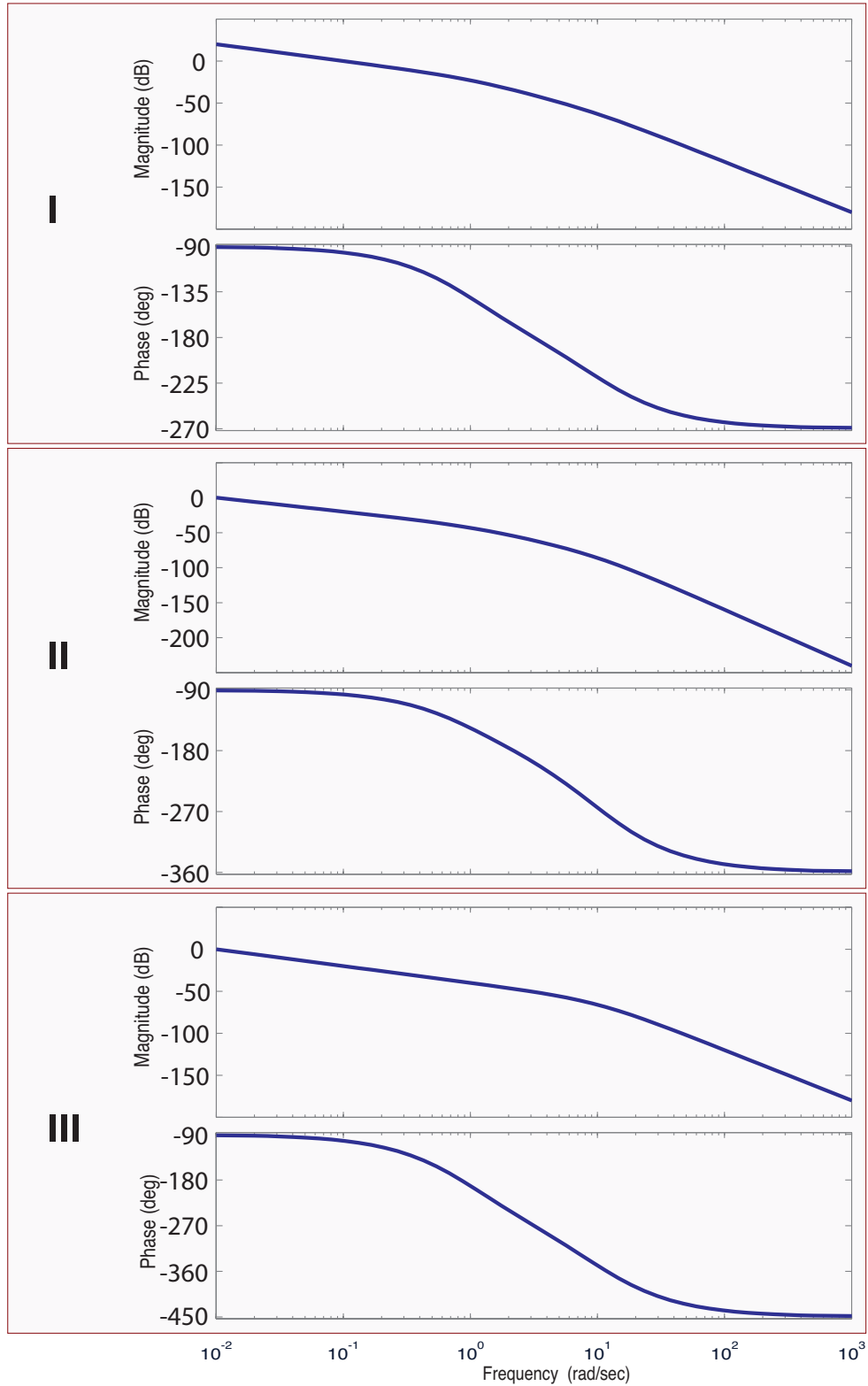Problem 3:   .....   / 20 points

Problem 4:   .....   / 20 points

Problem 5:   .....   / 15 points

TOTAL:   .....   / 85 points
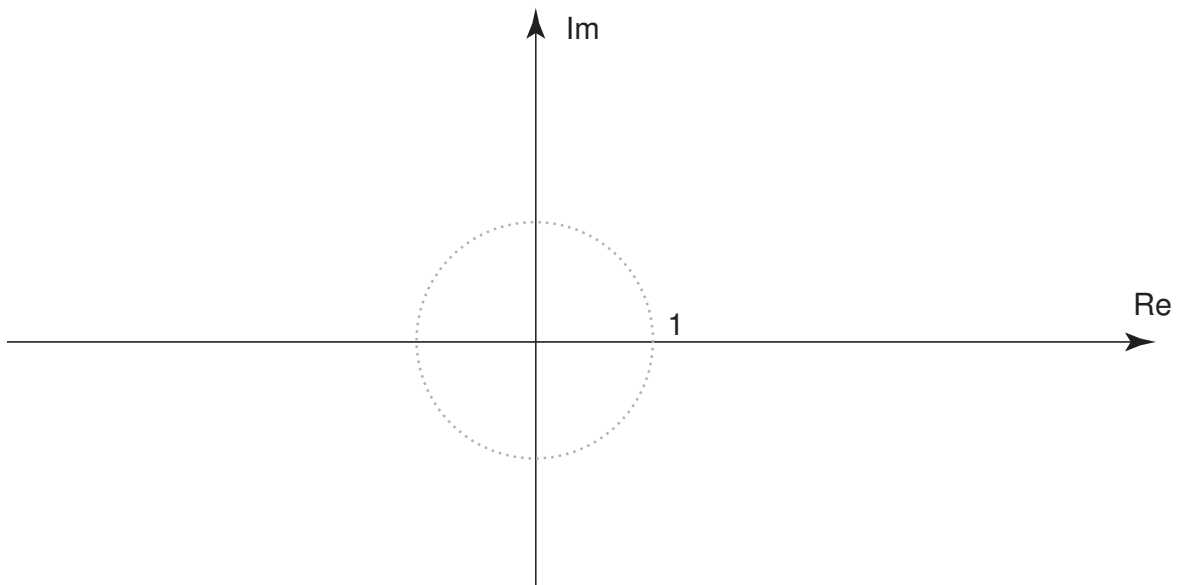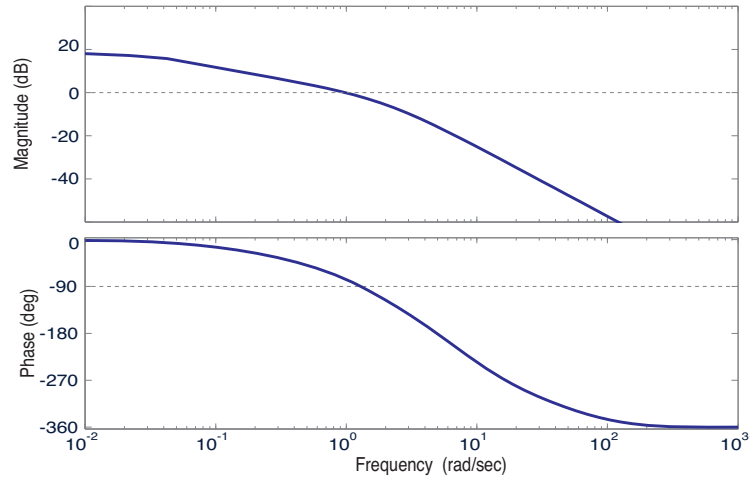
## Problem 1     (15 points)   *5+5+5*

**(a)** Indicate which of the Bode plots is valid for this transfer function, $G(s) = \dfrac{1}{s(s+1)(s+10)}$.

**(b)** For the Bode plot shown on the right, sketch a polar plot:

$$G(j\omega) \text{ for } \omega > 0$$

(the portion of the Nyquist plot corresponding to $\omega > 0$).

**(c)** Suppose that $G$ is any *stable* transfer function.

It is placed in the unity-feedback configuration shown on the front of this exam.

It is known that the closed loop system $Y/R$ is also stable.

Explain why the Nyquist plot for $G$ will show no encirclements of $-1$.

**Problem 2**     (15 points)   *6+3+3+3*

The transfer function $G = G_c G_p$ is
stable and minimum phase (all poles
and zeros in strict left half plane).
A polar plot is shown on the right:



The plot is labeled $G(j\omega)$ with Imaginary Axis and Real Axis.
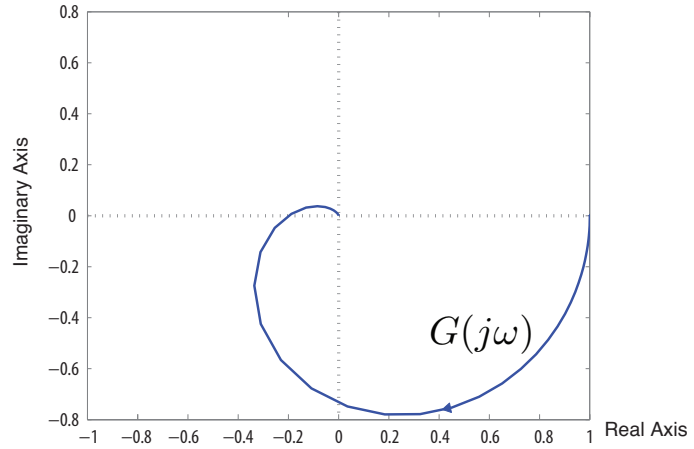
**(a)** Consider the unity-feedback closed loop system shown on the front of this exam.
Prove that the closed loop $Y/R$ is stable based on the Nyquist Stability Criterion.
*You must construct a complete Nyquist plot to justify your answer.*

**(b)** Estimate the gain margin of the closed loop system

**(c)** Estimate the phase margin of the closed loop system

**(d)** Estimate the steady-state error $e_\infty$ to a unit step reference input

**Problem 3**     (20 points)

The transfer function $G_p(s) = \dfrac{1}{s(s+1)}$ is the normalized model for a DC motor.

Design a compensator $G_c$ to obtain a settling time no greater than 0.1 sec., with less than 50% overshoot. *Note:* The settling time and the overshoot are denoted by $t_s$ and by $M_p$ on the front of this exam.

**Problem 4**     (20 points)

Consider the unity-feedback closed loop system shown on the front of this exam with

$$G_p(s) = \frac{1}{s(s+2)}.$$

Design a lead compensator, $G_c(s)$, so that the closed-loop system has a pair of poles at $s = -4$. *Hint:* try a pole-zero cancellation.

**Problem 5**     (15 points)

Consider the plant transfer function

$$G_p(s) = \frac{1}{s^2}$$

and a controller transfer function of the form

$$G_c(s) = K\frac{s+z}{s+3},$$

which is a lead or lag controller with the pole fixed at $p = 3$ but the zero and the feedback gain are free parameters to be chosen. Choose values for $z$ and $K$ for which the closed-loop system will have poles at $-3 \pm 3j$, or explain why this is not possible.

## Useful Facts

**Unilateral Laplace transforms:**

$$f(t), \ t \geq 0 \quad \xrightarrow{\mathscr{L}} \quad F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t, \ s \in \mathbb{C}$$

$$\mathscr{L}\left[f'(t)\right] = sF(s) - f(0)$$

$$\mathscr{L}\left[f''(t)\right] = s^2 F(s) - sf(0) - f'(0)$$

**Second-order system:**

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2} \qquad \omega_n, \zeta > 0$$

Rise time: $\quad t_r \approx \dfrac{1.8}{\omega_n}$

Peak time: $\quad t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

Overshoot: $\quad M_p = \exp\left(-\dfrac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

Settling time: $\quad t_s^{5\%} \approx \dfrac{3}{\zeta\omega_n}$

**Stability criteria for polynomials:**

- a monic polynomial $p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$ is *stable* if all of its roots are in the open LHP

- 2nd-order polynomial
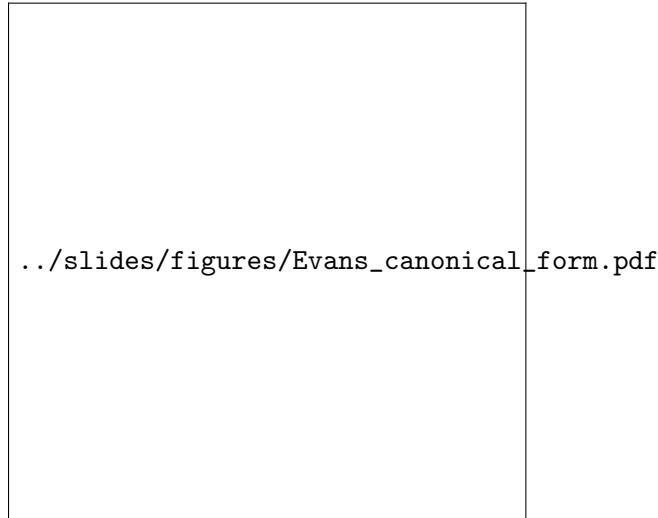$$p(s) = s^2 + a_1 s + a_2$$
is stable if and only if $a_1, a_2 > 0$

- 3rd-order polynomial
$$p(s) = s^3 + a_1 s^2 + a_2 s + a_3$$
is stable if and only if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$

**Root locus**  Let $L$ be a proper transfer function of the form

$$L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$



../slides/figures/Evans_canonical_form.pdf

The *root locus* is the set of all $s \in \mathbb{C}$ such that

$$1 + KL(s) = 0 \qquad \Longleftrightarrow \qquad a(s) + Kb(s) = 0$$

*Phase condition:* a point $s \in \mathbb{C}$ is on the RL if and only if

$$\angle L(s) = \angle \frac{b(s)}{a(s)} = \angle \frac{(s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)} = 180° \mod 360°$$

## Rules for sketching root loci

- Rule A: $n$ branches ($n = \#$(open-loop poles))

- Rule B: branches start at open-loop poles $p_1, \ldots, p_n$

- Rule C: $m$ of the branches end at open-loop zeros $z_1, \ldots, z_m$ ($L$ is proper: $m \leq n$)

- Rule D: a point $s \in \mathbb{R}$ is on the RL if and only if there is an *odd* number of *real* open-loop poles and zeros to the right of it

- Rule E: if $n - m > 0$, the remaining $n - m$ branches approach $\infty$ along asymptotes departing from the point

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n - m}$$

  at angles

$$\frac{(2\ell + 1) \cdot 180°}{n - m}, \qquad \ell = 0, 1, \ldots, n - m - 1.$$

- Rule F: $j\omega$-crossings

– find the critical value(s) of $K$ (if any) that will make the characteristic polynomial $a(s) + Kb(s)$ unstable

– for each of these critical values, solve

$$a(j\omega) + Kb(j\omega) = 0$$

for critical frequencies $\omega$

**Bode plots** A transfer function $G(j\omega)$ is in *Bode form* if it is written as a product of (some or all of) the following three types of factors:

- Type 1 — $n$th-order zero or pole at the origin, $K_0(j\omega)^n$, $K_0 > 0$, $n$ is an integer

- Type 2 — real zero or pole, $(j\omega\tau + 1)^{\pm 1}$, $\tau > 0$

- Type 3 — complex zero or pole, $\left[\left(\dfrac{j\omega}{\omega_n}\right)^2 + 2\zeta\dfrac{j\omega}{\omega_n} + 1\right]^{\pm 1}$, $\quad \omega_n > 0, \ 0 < \zeta < 1$

Magnitude and phase relationships:

|  | low frequency | real zero/pole | complex zero/pole |
|---|---|---|---|
| magnitude slope | $n$ | up/down by 1 | up/down by 2 |
| phase | $n \times 90°$ | up/down by 90° | up/down by 180° |

Crossover frequency: $|G(j\omega_c)| = 1$

**Bode plots for lead and lag compensators**

$$\text{lead: } D(s) = K\frac{s/z + 1}{s/p + 1}, \qquad z < p \quad \text{lag: } D(s) = \frac{s + z}{s + p}, \qquad z > p$$

**Stability margins** — assume $K$ is stabilizing

- **Gain Margin (GM)**: the factor by which $K$ has to be multiplied for the closed-loop system to become unstable

- **Phase Margin (PM)**: the amount by which the phase of $G(j\omega_c)$ differs from $\pm 180°$ (the sign depends on the magnitude slope of the Bode plot of $KG$)

**Nyquist plots**   For a transfer function $H(s)$, the Nyquist plot is the set of all points

$$\Big(\operatorname{Re} H(j\omega), \operatorname{Im} H(j\omega)\Big), \qquad -\infty < \omega < \infty$$

**The Argument Principle**      $N = Z - P$, where:

- $N = \#(\circlearrowright \text{ of } 0 \text{ by the Nyquist plot of } H)$

- $Z = \#(\text{RHP zeros of } H)$

- $P = \#(\text{RHP poles of } H)$

**Nyquist Stability Criterion**   — consider the unity feedback configuration:

Then $N = Z - P$, where:

- $N = \#(\circlearrowright \text{ of } -1/K \text{ by the Nyquist plot of } G)$

- $Z = \#(\text{RHP closed-loop poles})$

- $P = \#(\text{RHP open-loop poles})$

**Stability margins from Nyquist plots**

$$\text{GM} = \frac{1}{M_{180°}}, \qquad \text{PM} = \varphi$$