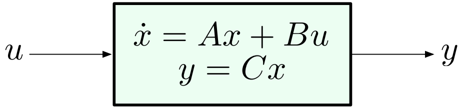
Plan of the Lecture

- Review: coordinate transformations; conversion of any controllable system to CCF.
- Today's topic: pole placement by (full) state feedback.

Goal: learn how to assign arbitrary closed-loop poles of a controllable system $\dot{x} = Ax + Bu$ by means of state feedback $u = -Kx$.

Reading: FPE, Chapter 7

State-Space Realizations



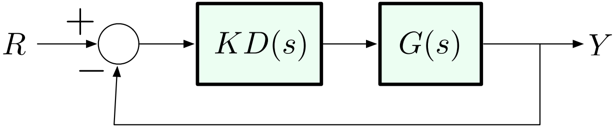
$$\downarrow$$

$$G(s) = C(Is - A)^{-1}B$$

Open-loop poles are the eigenvalues of A:

$$\det(Is - A) = 0$$

Then we add a controller to move the poles to desired locations:



Goal: Pole Placement by State Feedback

Consider a single-input system in state-space form:

$$u \longrightarrow \boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}} \longrightarrow y$$

Today, our goal is to establish the following fact:

If the above system is *controllable*, then we can assign arbitrary closed-loop poles by means of a state feedback law

$$\begin{aligned} u = -Kx &= -\begin{pmatrix} k_1 & k_2 & \dots & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= -(k_1 x_1 + \dots + k_n x_n), \end{aligned}$$

where K is a $1 \times n$ matrix of feedback gains.

Review: Controllability

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \,|\, AB \,|\, A^2B \,|\, \ldots \,|\, A^{n-1}B\right]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

- As we will see today, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

Coordinate Transformations

- We will see that state feedback design is particularly easy when the system is in CCF.
- Hence, we need a way of constructing a CCF state-space realization of a given controllable system.
- We will do this by suitably changing the coordinate system for the state vector.

Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \xrightarrow{T} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$

- The transfer function does not change.
- The controllability matrix is transformed:

$$\mathcal{C}(\bar{A}, \bar{B}) = T\mathcal{C}(A, B).$$

- The transformed system is controllable if and only if the original one is.
- If the original system is controllable, then

$$T = \mathcal{C}(\bar{A}, \bar{B}) \left[\mathcal{C}(A, B)\right]^{-1}.$$

This gives us a way of systematically passing to CCF.

Example: Converting a Controllable System to CCF

$$A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (C \text{ is immaterial})$$

Step 1: check for controllability.

$$\mathcal{C} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix} \qquad \det \mathcal{C} = -1 \qquad \text{– controllable}$$

Step 2: Determine desired $\mathcal{C}(\bar{A}, \bar{B})$.

$$\mathcal{C}(\bar{A}, \bar{B}) = [\bar{B} \,|\, \bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

Step 3: Compute T.

$$T = \mathcal{C}(\bar{A}, \bar{B}) \cdot [\mathcal{C}(A, B)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Finally, Pole Placement via State Feedback

Consider a state-space model

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}$$
$$y = x$$

Let's introduce a *state feedback law*

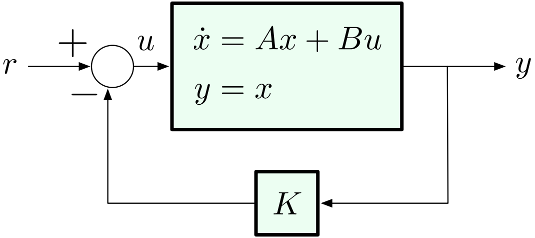
$$\begin{aligned} u &= -Ky \equiv -Kx \\ &= -\begin{pmatrix} k_1 & k_2 & \dots & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = -(k_1 x_1 + \dots + k_n x_n) \end{aligned}$$

Closed-loop system:

$$\dot{x} = Ax - BKx = (A - BK)x$$
$$y = x$$

Pole Placement via State Feedback

Let's also add a reference input:



$$\dot{x} = Ax + B(-Kx + r) = (A - BK)x + Br, \qquad y = x$$

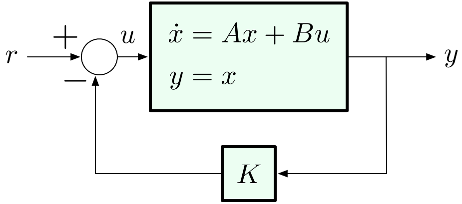
Take the Laplace transform:

$$sX(s) = (A - BK)X(s) + BR(s), \; Y(s) = X(s)$$

$$Y(s) = \underbrace{(Is - A + BK)^{-1}B}_{G} R(s)$$

Closed-loop poles are the eigenvalues of $A - BK$!!

Pole Placement via State Feedback



assigning closed-loop poles = assigning eigenvalues of $A - BK$

Now we will see that this is particularly straightforward if the (A, B) system is in CCF.

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

The Beauty of CCF

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Claim.

$$\det(Is - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

— the last row of the A matrix in CCF consists of the coefficients of the characteristic polynomial, in reverse order, with "−" signs.

Proof of the Claim

A nice way is via Laplace transforms:

$$\dot{x} = Ax + Bu$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Represent this as a system of ODEs:

$$\begin{aligned} \dot{x}_1 &= x_2 & X_2 &= sX_1 \\ \dot{x}_2 &= x_3 & X_3 &= sX_2 = s^2 X_1 \\ &\vdots & &\vdots \\ \dot{x}_n &= -\sum_{i=1}^{n} a_{n-i+1} x_i + u & \underbrace{\left(s^n + a_1 s^{n-1} + \ldots + a_n\right)}_{\text{char. poly.}} X_1 &= U \end{aligned}$$

... And, Back to Pole Placement

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}$$

$$BK = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & \dots & k_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ k_1 & k_2 & k_3 & \dots & k_{n-1} & k_n \end{pmatrix}$$

$$A - BK = -\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & a_{n-2} + k_3 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

— still in CCF!!

Pole Placement in CCF

$$\dot{x} = (A - BK)x + Br, \qquad y = Cx$$

$$A - BK = -\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

Closed-loop poles are the roots of the characteristic polynomial

$$\begin{aligned} &\det(Is - A + BK) \\ &= s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_{n-1} + k_2)s + (a_n + k_1) \end{aligned}$$

Key observation: When the system is in CCF, each control gain affects only *one* of the coefficients of the characteristic polynomial, and these coefficients can be assigned arbitrarily by a suitable choice of $k_1, \ldots, k_n$.

Hence the name Controller Canonical Form — convenient for control design.

Pole Placement by State Feedback

General procedure for any *controllable* system:

1. Convert to CCF using a suitable invertible coordinate transformation T (such a transformation exists by controllability).
2. Solve the pole placement problem in the new coordinates.
3. Convert back to original coordinates.

Example

Given $\dot{x} = Ax + Bu$

$$A = \begin{pmatrix} -15 & 8 \\ -7 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Goal: apply state feedback to place closed-loop poles at $-10 \pm j$.

Step 1: convert to CCF — already did this

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \longrightarrow \qquad \bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \ \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example

Step 2: find $u = -\bar{K}\bar{x}$ to place closed-loop poles at $-10 \pm j$.

Desired characteristic polynomial:

$$(s + 10 + j)(s + 10 - j) = (s + 10)^2 + 1 = s^2 + 20s + 101$$

Thus, the closed-loop system matrix should be

$$\bar{A} - \bar{B}\bar{K} = \begin{pmatrix} 0 & 1 \\ -101 & -20 \end{pmatrix}$$

On the other hand, we know

$$\bar{A} - \bar{B}\bar{K} = \begin{pmatrix} 0 & 1 \\ -(15 + \bar{k}_1) & -(8 + \bar{k}_2) \end{pmatrix} \quad \Longrightarrow \quad \bar{k}_1 = 86,\ \bar{k}_2 = 12$$

This gives the control law

$$u = -\bar{K}\bar{x} = -\begin{pmatrix} 86 & 12 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

Example

Step 3: convert back to the old coordinates.

$$\begin{aligned} u &= -\bar{K}\bar{x} \\ &= -\underbrace{\bar{K}T}_{K} x \end{aligned}$$

— therefore,

$$\begin{aligned} K &= \bar{K}T \\ &= \begin{pmatrix} 86 & 12 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 86 & -74 \end{pmatrix} \end{aligned}$$

The desired state feedback law is

$$u = \begin{pmatrix} -86 & 74 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$