Plan of the Lecture

- Review: frequency-domain design method.
- Today's topic: introduction to state-space design.

Goal: introduce basic notions of state-space control: different state-space realizations of the same transfer function; several canonical forms of state-space systems; controllability matrix.

Reading: FPE, Chapter 7

Frequency-Domain vs. State-Space

- 90% of industrial controllers are designed using frequency-domain methods (PID is a popular architecture)
- 90% of current research in systems and control is in the state-space framework

To be able to talk to control engineers *and* follow progress in the field, we need to know both methods and *understand the connections between them.*

State-Space Methods

- the state-space approach reveals *internal system architecture* for a given transfer function
- the mathematics is different: heavy use of *linear algebra*
- this is just a short introduction; to learn this material properly, take ECE 515

A General State-Space Model

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

$$\text{output } y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where:

A – system matrix $(n \times n)$

B – input matrix $(n \times m)$

C – output matrix $(p \times n)$

D – feedthrough matrix $(p \times m)$

From State-Space to Transfer Function

Let us find the *transfer function* from u to y corresponding to the state-space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- in the scalar case ($x, y, u \in \mathbb{R}$), we took the Laplace transform
- the same idea here when working with vectors: just do it component by component

From State-Space to Transfer Function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$$

Recall matrix-vector multiplication:

$$\begin{aligned} \dot{x}_i &= (Ax)_i + (Bu)_i \\ &= \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k \end{aligned}$$

$$\begin{aligned} y_\ell &= (Cx)_\ell + (Du)_\ell \\ &= \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k \end{aligned}$$

From State-Space to Transfer Function

Now we take the Laplace transform:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k$$

$$\downarrow \mathscr{L}$$

$$sX_i(s) - x_i(0) = \sum_{j=1}^{n} a_{ij} X_j(s) + \sum_{k=1}^{m} b_{ik} U_k(s), \qquad i = 1, \ldots, n$$

Write down in matrix-vector form:

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(Is - A)X(s) = x(0) + BU(s) \qquad (I \text{ is the } n \times n \text{ identity matrix})$$

$$X(s) = (Is - A)^{-1} x(0) + (Is - A)^{-1} BU(s)$$

From State-Space to Transfer Function

$$y_\ell = \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k$$

$$\downarrow \mathscr{L}$$

$$Y_\ell(s) = \sum_{j=1}^{n} c_{\ell j} X_j(s) + \sum_{k=1}^{m} d_{\ell k} U_k(s), \qquad \ell = 1, \ldots, p$$

Write down in matrix-vector form:

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C\left[(Is - A)^{-1}x(0) + (Is - A)^{-1}BU(s)\right] + DU(s) \\ &= C(Is - A)^{-1}x(0) + \left[C(Is - A)^{-1}B + D\right]U(s) \end{aligned}$$

To find the input-output t.f., set the IC to 0:

$$Y(s) = G(s)U(s), \qquad \text{where } G(s) = C(Is - A)^{-1}B + D$$

From State-Space to Transfer Function

The transfer function from u to y, corresponding to

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is given by

$$G(s) = C(Is - A)^{-1}B + D$$

Observe that $G(s)$ contains information about the state-space matrices A, B, C, D!!

From State-Space to Transfer Function

$$\dot{x} = Ax + Bu \qquad\qquad Y(s) = G(s)U(s)$$
$$y = Cx + Du \qquad\qquad\qquad = \left[C(Is - A)^{-1}B + D\right]U(s)$$

Important!!

- $G(s)$ is *undefined* when the $n \times n$ matrix $Is - A$ is *singular* (or noninvertible), i.e., precisely when $\det(Is - A) = 0$
- since A is $n \times n$, $\det(Is - A)$ is a *polynomial* of degree n (the characteristic polynomial of A):

$$\det(Is - A) = \det \begin{pmatrix} s - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & s - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & s - a_{nn} \end{pmatrix},$$

  and its roots are the eigenvalues of A
- G is (open-loop) stable if all eigenvalues of A lie in LHP.

Example: Computing $G(s)$

Consider the state-space model in Controller Canonical Form (CCF)*:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

— this is a *single-input, single-output* (SISO) system, since $u, y \in \mathbb{R}$; the state is two-dimensional.

Let's compute the transfer function:

$$G(s) = C(Is - A)^{-1}B \qquad\qquad (D = 0 \text{ here})$$

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix}$$

* We will explain this terminology later.

Example: Computing $G(s)$

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix}$$ — how do we compute $(Is - A)^{-1}$?

A useful formula for the inverse of a 2×2 matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M \neq 0 \quad \Longrightarrow \quad M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Applying the formula, we get

$$\begin{aligned} (Is - A)^{-1} &= \frac{1}{\det(Is - A)} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \\ &= \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \end{aligned}$$

Example: Computing $G(s)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} G(s) &= C(Is - A)^{-1}B \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{s^2 + 5s + 6} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix} \\ &= \frac{s+1}{s^2 + 5s + 6} \end{aligned}$$

- the above state-space model is a *realization* of this t.f.
- note how coefficients 5 and 6 appear in both $G(s)$ and A!!

State-Space Realizations of Transfer Functions

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$G(s) = \frac{s+1}{s^2 + 5s + 6}$$

— at least in this example, information about the state-space model (A, B, C) is contained in $G(s)$.

Is this information *recoverable*? — i.e., is there only one state-space realization of a given t.f.? Or are there many?

Answer: There are infinitely many!

State-Space Realizations of Transfer Functions

Start with

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and consider a new state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}, \quad \bar{B} = C^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{C} = B^T = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

This is a different state-space model!

State-Space Realizations of Transfer Functions

Claim: The state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T$$

has the same transfer function as the original model with (A, B, C).

Proof:

$$\begin{aligned}
\bar{C}(Is - \bar{A})^{-1}\bar{B} &= B^T \left(Is - A^T\right)^{-1} C^T \\
&= B^T \left[(Is - A)^T\right]^{-1} C^T \\
&= B^T \left[(Is - A)^{-1}\right]^T C^T \\
&= \left[C(Is - A)^{-1}B\right]^T \\
&= C(Is - A)^{-1}B
\end{aligned}$$

State-Space Realizations of Transfer Functions

The state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad\qquad y = \bar{C}x$$

with

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T$$

has the same transfer function as the original model with (A, B, C).

But the state-space model is now in the Observer Canonical Form (OCF):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Even More Realizations ...

Yet another realization of $G(s) = \dfrac{s+1}{s^2+5s+6}$ can be extracted from the partial-fractions decomposition:

$$G(s) = \frac{s+1}{(s+2)(s+3)} = \frac{2}{s+3} - \frac{1}{s+2}.$$

This is the Modal Canonical Form (MCF):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then
$$\begin{aligned} C(Is-A)^{-1}B &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} s+3 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} \\ \frac{1}{s+2} \end{pmatrix} = \frac{2}{s+3} - \frac{1}{s+2} \end{aligned}$$

State-Space Realizations: Bottom Line

- a given transfer function $G(s)$ can be realized using infinitely many state-space models
- certain properties make some realizations preferable to others
- one such property is *controllability*

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B \right]$$

— recall that A is $n \times n$ and B is $n \times 1$, so $\mathcal{C}(A, B)$ is $n \times n$;
— the controllability matrix only involves A and B, not C

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

(This definition is only true for the single-input case; the multiple-input case involves the *rank* of $\mathcal{C}(A, B)$.)

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \,|\, AB \,|\, A^2 B \,|\, \dots \,|\, A^{n-1} B \right]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Example: Computing $\mathcal{C}(A, B)$

Let's get back to our old friend:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2\times 2} \implies \mathcal{C}(A, B) \in \mathbb{R}^{2\times 2}$

$$\mathcal{C}(A, B) = [B \,|\, AB] \qquad AB = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\implies \mathcal{C}(A, B) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

Is this system controllable?

$$\det \mathcal{C} = -1 \neq 0 \qquad \implies \qquad \text{system is controllable}$$

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

CCF with Arbitrary Zeros

In our example, we had $G(s) = \dfrac{s+1}{s^2+5s+6}$, with a minimum-phase zero at $z = -1$.

Let's consider a general zero location $s = z$:

$$G(s) = \frac{s-z}{s^2+5s+6}$$

This gives us a CCF realization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since A, B are the same, $\mathcal{C}(A, B)$ is the same $\Longrightarrow$ the system is still controllable.

A system in CCF is controllable for any locations of the zeros.