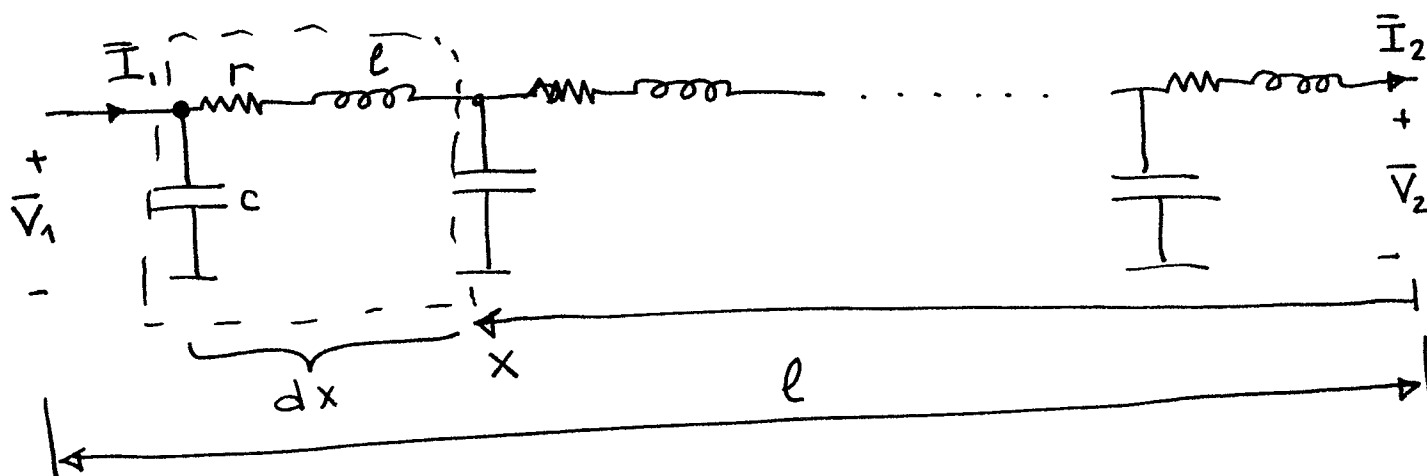


LECTURE 9

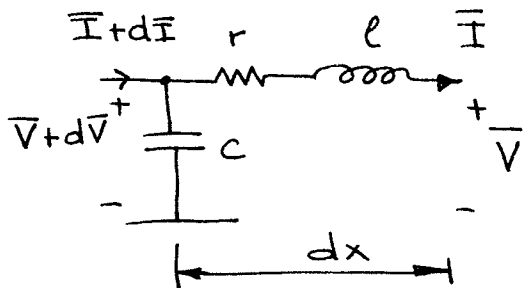
- DERIVATION OF TERMINAL V, I RELATIONS
- TRANSMISSION MATRIX MODEL
- LUMPED-CIRCUIT EQUIVALENT
- SIMPLIFIED MODELS
- POWER TRANSFER IN SHORT LINES

DERIVATION OF TERMINAL V, I RELATIONS

- In previous lectures, we spent time deriving per-unit length inductance and capacitance of transmission lines.
- We will now use these parameters to develop transmission line models.



Let's focus on a differential line length:



$$\begin{aligned} \bar{Z} &= r + j\omega l \\ \bar{Y} &= j\omega c \end{aligned} \left\{ \begin{array}{l} \rightarrow \text{These are} \\ \text{per-unit length} \\ \text{quantities.} \end{array} \right.$$

Remember: In steady-state operation, all quantities are phasors & impedances

Let's write Kirchhoff's laws:

$$\bar{V} + d\bar{V} = \bar{V} + (r + j\omega l) \bar{I} \stackrel{dx}{=} \bar{V} + \bar{Z} dx \bar{I}$$

$$d\bar{I} = \bar{Y} dx (\bar{V} + d\bar{V})$$

$$d\bar{V} = \bar{Z} dx \cdot \bar{I} \rightarrow \frac{d\bar{V}}{dx} = \bar{Z} \cdot \bar{I}$$

$$d\bar{I} = \bar{Y} dx \cdot \bar{V} + \bar{Y} dx \cdot d\bar{V} \simeq \bar{Y} dx \bar{V}$$

$$\rightarrow \left. \begin{aligned} \frac{d\bar{V}}{dx} &= \bar{Z} \bar{I} \\ \frac{d\bar{I}}{dx} &= \bar{Y} \bar{V} \end{aligned} \right\} \rightarrow \underbrace{\begin{bmatrix} \frac{d\bar{V}}{dx} \\ \frac{d\bar{I}}{dx} \end{bmatrix}}_{\text{We need to solve this set of diff. eqns.}} = \begin{bmatrix} 0 & \bar{Z} \\ \bar{Y} & 0 \end{bmatrix} \begin{bmatrix} \bar{V} \\ \bar{I} \end{bmatrix}$$

We need to solve this set of diff. eqns.

Define $\gamma \triangleq \sqrt{\bar{Z} \cdot \bar{Y}}$ (it is a complex quantity)

Called propagation constant.

If $r = 0 \rightarrow \boxed{\gamma = j\omega \sqrt{lc}}$ $\rightarrow$ purely imaginary

The solution is:

$$\begin{bmatrix} \bar{V}(x) \\ \bar{I}(x) \end{bmatrix} = \begin{bmatrix} \frac{e^{\gamma x} + e^{-\gamma x}}{2} & \sqrt{\frac{\bar{Z}}{\bar{Y}}} \frac{(e^{\gamma x} - e^{-\gamma x})}{2} \\ \sqrt{\frac{\bar{Y}}{\bar{Z}}} \frac{(e^{\gamma x} - e^{-\gamma x})}{2} & \frac{e^{\gamma x} - e^{-\gamma x}}{2} \end{bmatrix} \begin{bmatrix} \bar{V}(0) \\ \bar{I}(0) \end{bmatrix}$$

↑
The voltage and current measured @ any point x units of length away from the "receiving end" i.e., $x=0$ ($\bar{V}_2, \bar{I}_2$)

$$\bar{V}(0) = \bar{V}_2, \bar{I}(0) = \bar{I}_2, \bar{V}(x=l) = \bar{V}_1, \bar{I}(x=l) = \bar{I}_1$$

$$\begin{bmatrix} \bar{V}_1 \\ \bar{I}_1 \end{bmatrix} = \begin{bmatrix} \frac{e^{\gamma l} + e^{-\gamma l}}{2} & \sqrt{\frac{\bar{Z}}{\bar{Y}}} \frac{(e^{\gamma l} - e^{-\gamma l})}{2} \\ \sqrt{\frac{\bar{Y}}{\bar{Z}}} \frac{(e^{\gamma l} - e^{-\gamma l})}{2} & \frac{e^{\gamma l} - e^{-\gamma l}}{2} \end{bmatrix} \begin{bmatrix} \bar{V}_2 \\ \bar{I}_2 \end{bmatrix}$$

Characteristic impedance: ($\bar{Z}_c$)

$$\bar{Z}_c = \sqrt{\frac{\bar{Z}}{\bar{Y}}}$$

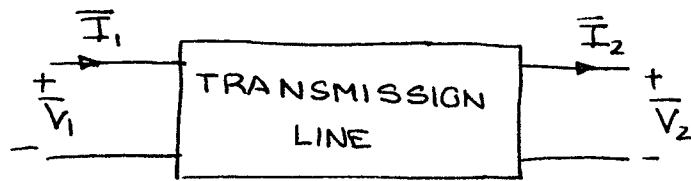
$$\frac{e^{\gamma l} + e^{-\gamma l}}{2} = \cosh \gamma l$$

$$\frac{e^{\gamma l} - e^{-\gamma l}}{2} = \sinh \gamma l$$

$$\begin{bmatrix} \bar{V}_1 \\ \bar{I}_1 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & \bar{Z}_c \sinh \gamma l \\ \frac{1}{\bar{Z}_c} \sinh \gamma l & \cosh \gamma l \end{bmatrix} \begin{bmatrix} \bar{V}_2 \\ \bar{I}_2 \end{bmatrix}$$

TRANSMISSION MATRIX OR ABCD MODEL

We can think of the transmission line as a "blackbox" where we are only interested in the terminal characteristics.



$$\begin{bmatrix} \bar{V}_1 \\ \bar{I}_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{V}_2 \\ \bar{I}_2 \end{bmatrix}$$

Transmission matrix: T

From the previous analysis, it is easy to see that

$$A = \cosh \gamma l$$

$$C = \frac{1}{Z_c} \sinh \gamma l$$

$$B = Z_c \sinh \gamma l$$

$$D = \cosh \gamma l$$

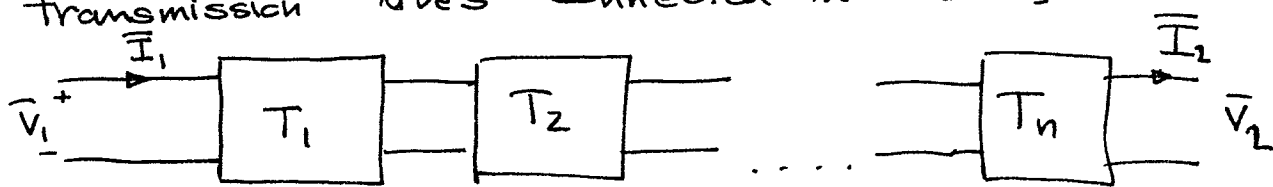
$$T = \begin{bmatrix} \cosh \gamma l & Z_c \sinh \gamma l \\ \frac{1}{Z_c} \sinh \gamma l & \cosh \gamma l \end{bmatrix}$$

Gives the "sending end" voltage and current as a function of voltage and current in the "receiving end"

$$T^{-1} = \begin{bmatrix} \cosh \gamma l & -Z_c \sinh \gamma l \\ -\frac{1}{Z_c} \sinh \gamma l & \cosh \gamma l \end{bmatrix}$$

Gives the voltage and current in the "receiving end" as a function of voltage and current in the "sending end"

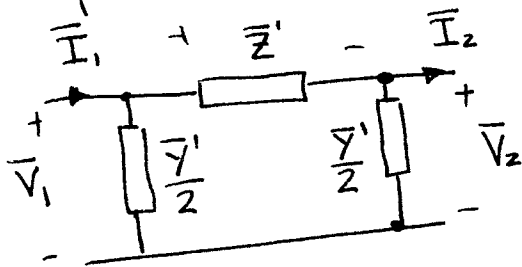
Transmission matrices are an interesting notion as they can be used to easily develop models of transmission lines connected in series



$$\begin{bmatrix} \bar{V}_1 \\ \bar{I}_1 \end{bmatrix} = T_1 \cdot T_2 \cdot \dots \cdot T_n \cdot \begin{bmatrix} \bar{V}_2 \\ \bar{I}_2 \end{bmatrix}$$

LUMPED-CIRCUIT MODEL (EQUIVALENT π CIRCUIT)

- Instead of using the transmission matrix model, we are interested in deriving a lumped-circuit equivalent model.



$$\begin{bmatrix} \bar{V}_1 \\ \bar{I}_1 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & \bar{Z}_c \sinh \gamma l \\ \frac{1}{\bar{Z}_c} \sinh \gamma l & \cosh \gamma l \end{bmatrix} \begin{bmatrix} \bar{V}_2 \\ \bar{I}_2 \end{bmatrix}$$

Kirchhoff's equations:

$$\bar{V}_1 = \bar{V}_2 + \left(\bar{I}_2 + \frac{\bar{Y}'}{2} \bar{V}_2 \right) \bar{Z}' \rightarrow$$

$$\rightarrow \bar{V}_1 = \left(1 + \frac{\bar{Y}' \cdot \bar{Z}'}{2} \right) \bar{V}_2 + \bar{Z}' \bar{I}_2$$

$$\bar{Z}_c = \sqrt{\frac{\bar{Z}}{\bar{Y}}}$$

$$\left(1 + \frac{\bar{Y}' \cdot \bar{Z}'}{2} \right) = \cosh \gamma l$$

$$\gamma = \sqrt{\bar{Z} \cdot \bar{Y}}$$

$$\bar{Z}' = \bar{Z}_c \sinh \gamma l$$

$$i) \bar{Z}' = \sqrt{\frac{\bar{Z}}{\bar{Y}}} \cdot \left[\frac{\gamma l}{\gamma l} \right] \sinh \gamma l = \sqrt{\frac{\bar{Z}}{\bar{Y}}} \sqrt{\bar{Z} \bar{Y}} l \cdot \frac{\sinh \gamma l}{\gamma l} = (\bar{Z} \cdot l) \frac{\sinh \gamma l}{\gamma l}$$

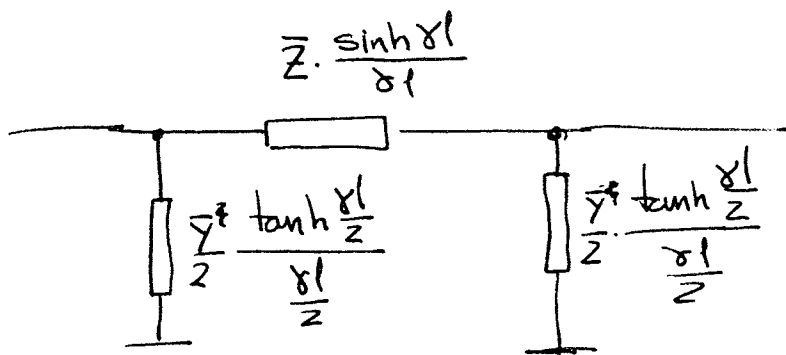
$$\boxed{\bar{Z}' = (\bar{Z} \cdot l) \cdot \frac{\sinh \gamma l}{\gamma l}}$$

$$ii) \frac{\bar{Y}'}{2} = \frac{1}{\bar{Z}'} (\cosh \gamma l - 1) = \frac{\cosh \gamma l - 1}{\bar{Z}_c \sinh \gamma l} = \frac{1}{\bar{Z}_c} \tanh \frac{\gamma l}{2}$$

As before:

$$\frac{\bar{Y}'}{2} = \frac{1}{\bar{Z}} \left[\frac{\gamma l}{2} \right] \tanh \frac{\gamma l}{2} = \frac{\gamma l}{\bar{Z}} \frac{\tanh \frac{\gamma l}{2}}{\frac{\gamma l}{2}}$$

$$\boxed{\frac{\bar{Y}'}{2} = \frac{\bar{Y}}{\bar{Z}} \frac{\tanh \frac{\gamma l}{2}}{\frac{\gamma l}{2}}}$$



$$\bar{Z} = Z \cdot l$$

$$\bar{Y} = Y \cdot l$$

SIMPLIFIED MODELS

length l
 $l > 200$ miles

$$\bar{Z}' = \bar{Z} \cdot \frac{\sinh \gamma l}{\gamma l}$$

$$\frac{\bar{Y}'}{\bar{Z}} = \frac{Y}{Z} \frac{\tanh \frac{\gamma l}{2}}{\frac{\gamma l}{2}}$$

$50 < l < 200$

$$\bar{Z}' = \bar{Z}$$

$$\bar{Y}' = \frac{Y}{2}$$

$l < 50$

$$\bar{Z}' = \bar{Z}$$

$$\bar{Y}' = 0$$

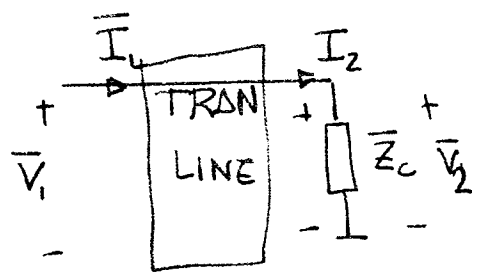
LOSELESS TRANSMISSION LINES

$$r=0 \rightarrow \bar{Z}_c = \sqrt{\frac{j\omega l}{j\omega c}} = \sqrt{\frac{l}{c}} = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \Omega$$

- For lossless transmission lines, the characteristic impedance is referred to as surge impedance

Surge impedance loading (SIL)

It is the power delivered by a lossless line to a load resistance equal to the surge impedance.



$$\begin{aligned}\bar{V}(x) &= \bar{V}_2 \cosh \gamma x + \bar{Z}_c \bar{I}_2 \sinh \gamma x \\ \bar{I}(x) &= \frac{\bar{V}_2}{\bar{Z}_c} \sinh \gamma x + \cosh \gamma x \bar{I}_2\end{aligned}$$

$$\bar{V}(x) = \bar{V}_2 (\cosh \gamma x + \sinh \gamma x) = \bar{V}_2 e^{\gamma x}$$

$$\bar{I}(x) = \bar{I}_2 (\cosh \gamma x + \sinh \gamma x) = \bar{I}_2 e^{\gamma x}$$

$$\gamma = j\omega\sqrt{LC}$$

$$|\bar{V}(x)| = |\bar{V}_2| = |\bar{V}_1|$$

$$|\bar{I}(x)| = |\bar{I}_2| = |\bar{I}_1|$$

- The voltage profile remains constant!

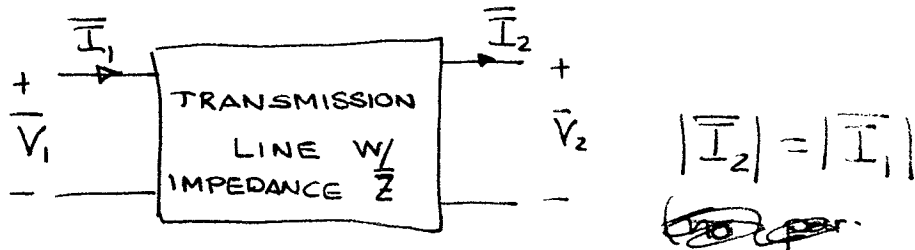
$$-\bar{S}(x) = \bar{V}(x) \cdot \bar{I}(x)^* = \bar{V}_2 \cdot \bar{I}_2^* = \bar{V}_1 \bar{I}_1^*$$

However, remember that $\bar{I}_2 = \frac{\bar{V}_2}{\bar{Z}_c} = \frac{\bar{V}_2}{\sqrt{\frac{L}{C}}}$

$$\Rightarrow \boxed{\bar{S}(x) = \frac{|\bar{V}_2|^2}{\bar{Z}_c}} \rightarrow \text{real!!}$$

- The complex power is real, it means there is no reactive power flow across the line
- The real power also remains constant along the line
- The voltage profile remains constant.

POWER TRANSFER IN SHORT-LINES



$$\bar{S}_{12} = \bar{V}_1 \cdot \bar{I}_1^* = \bar{V}_1 \cdot \left(\frac{\bar{V}_1 - \bar{V}_2}{\bar{Z}} \right)^*$$

$$\bar{V}_1 = V_1 \angle \theta_1, \quad \bar{V}_2 = V_2 \angle \theta_2, \quad \bar{Z} = Z \angle \theta_z$$

$$\begin{aligned} \bar{S}_{12} &= V_1 \angle \theta_1 \cdot \left(\frac{V_1 \angle -\theta_1 - V_2 \angle -\theta_2}{Z \angle -\theta_z} \right)^* \\ &= \frac{|V_1|^2}{Z} \angle \theta_z - \frac{V_1 V_2}{Z} \angle \theta_1 - \theta_2 + \theta_z \end{aligned}$$

If the line is lossless: $\theta_z = \frac{\pi}{2}$

$$\bar{S}_{12} = \frac{|V_1|^2}{Z} \angle \frac{\pi}{2} - \frac{V_1 V_2}{Z} \angle \theta_1 - \theta_2 + \frac{\pi}{2}$$

$$\begin{aligned} P_{12} &= \text{Re} \{ \bar{S}_{12} \} = \frac{V_1 V_2}{Z} \sin(\theta_1 - \theta_2) \\ Q_{12} &= \text{Im} \{ \bar{S}_{12} \} = \frac{|V_1|^2}{Z} - \frac{V_1 V_2}{Z} \cos(\theta_1 - \theta_2) \end{aligned}$$