ECE 463: Digital Communications Lab.

Lecture 6: Carrier Frequency Offset

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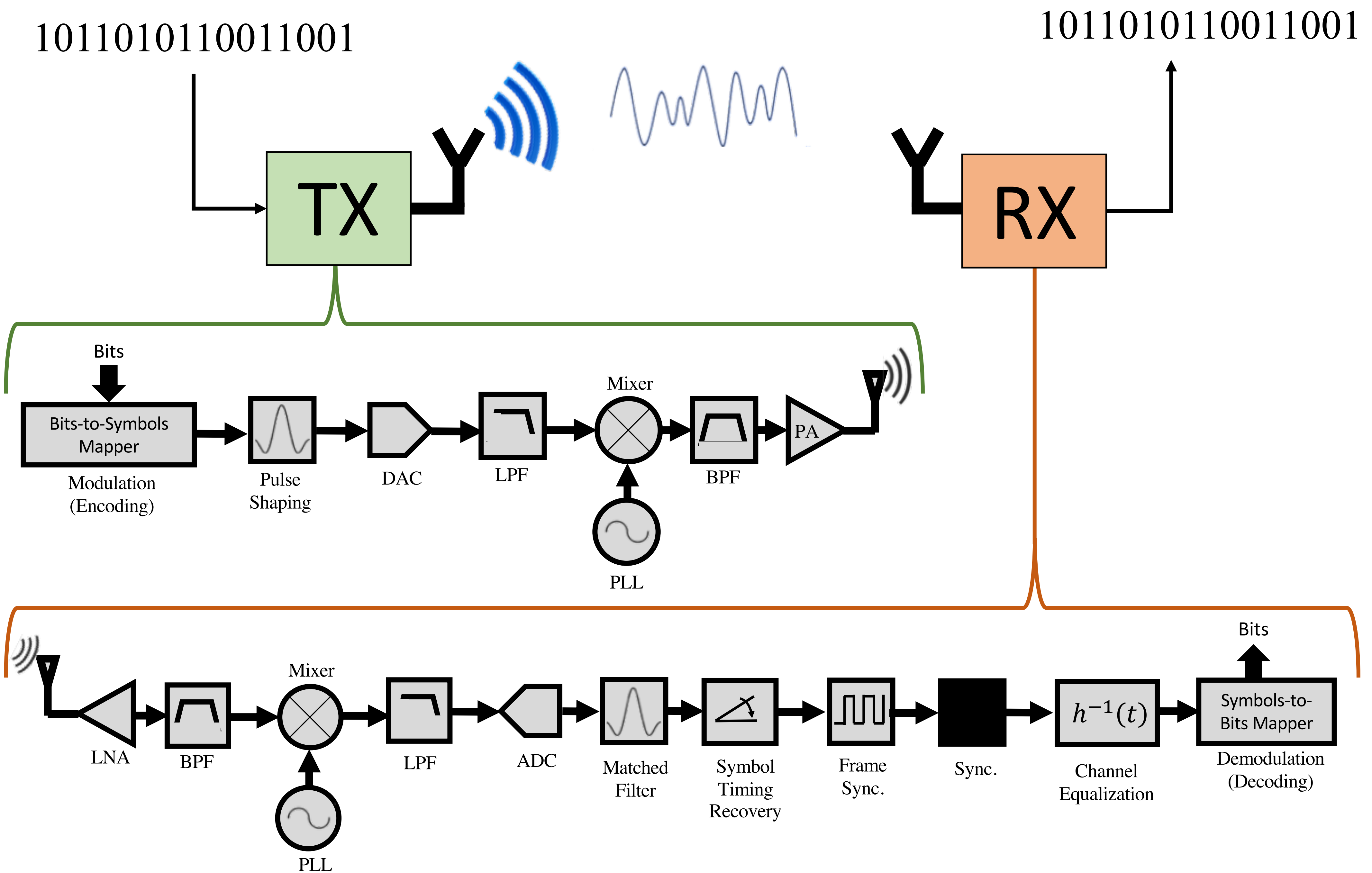
Previous Lecture:

- ✓ Multipath Channel
- ✓ Channel Estimation & Correction
- ✓ Narrowband vs. Wideband Channels
- ✓ Channel Equalization

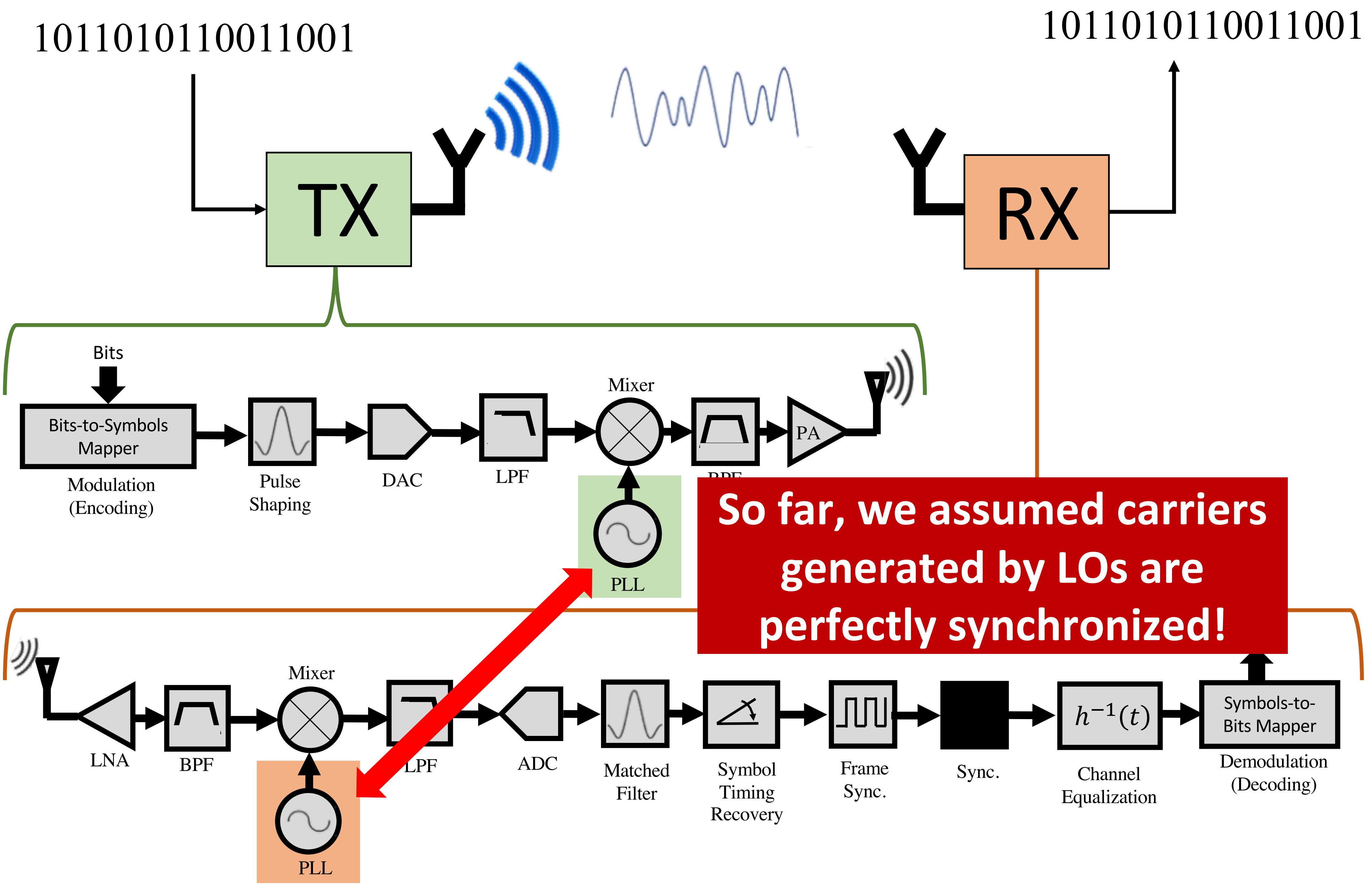
This Lecture:

- ❑ Carrier Frequency Offset Estimation & Correction
- ❑ Carrier Recovery & Phase Tracking

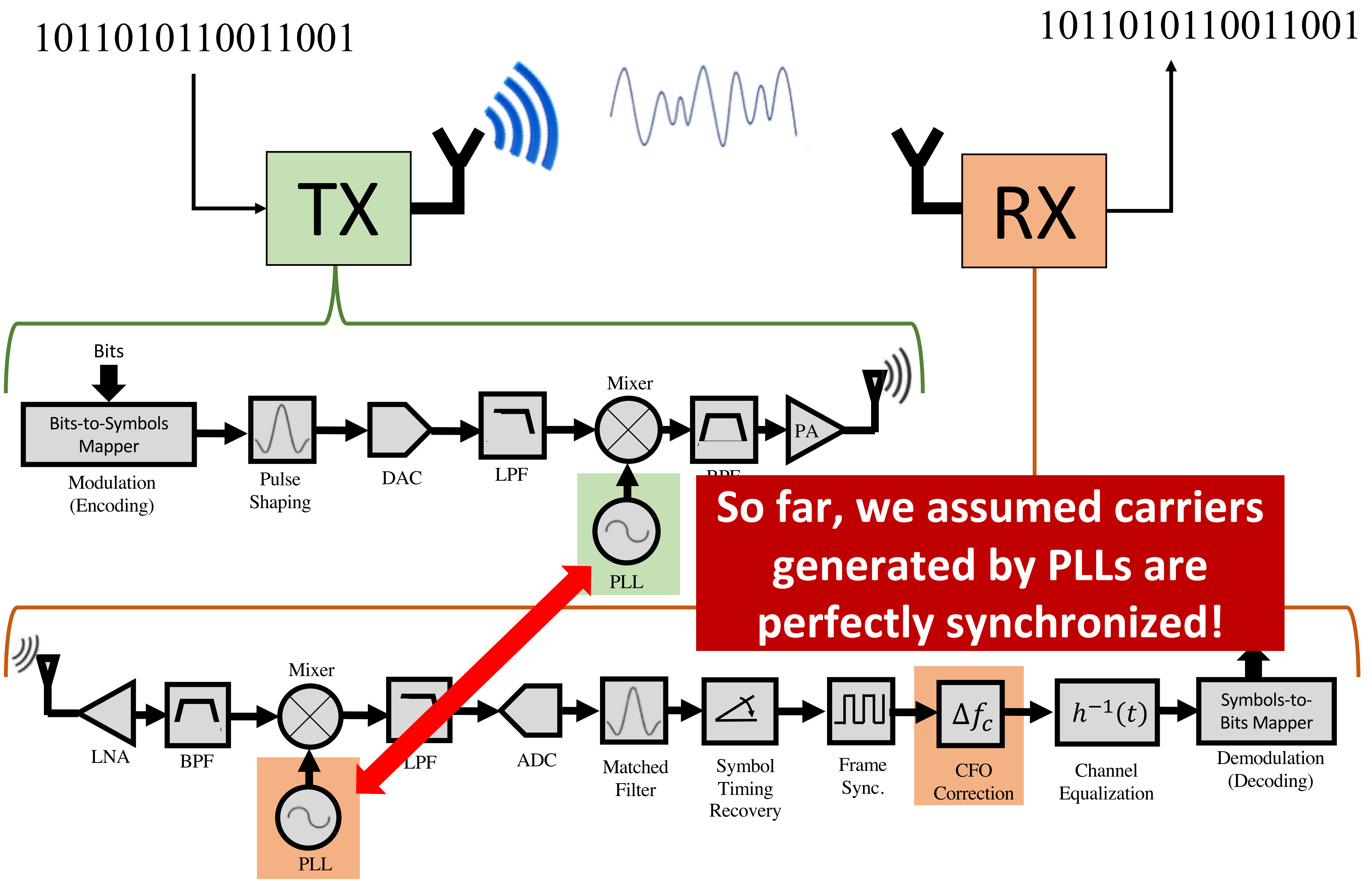
Digital Communication System



Digital Communication System



Digital Communication System



So far, we assumed carriers generated by PLLs are perfectly synchronized!

Carrier Frequency Offset

$$x(t) \Rightarrow x(t) \times e^{-j2\pi f_c t} \Rightarrow \alpha\, x(t-\tau) e^{-j2\pi f_c (t-\tau)}$$

Carrier Frequency Offset

$$x(t) \Rightarrow x(t) \times e^{-j2\pi f_c t} \Rightarrow \alpha\, e^{j2\pi f_c \tau}\, x(t-\tau) e^{-j2\pi f_c t}$$

Carrier Frequency Offset

$x(t)$ ➡ $x(t) \times e^{-j2\pi f_c t}$ ➡ $h\, x(t-\tau) e^{-j2\pi f_c t}$ ➡ $h\, x(t-\tau) e^{-j2\pi f_c t} \times e^{j2\pi f_c t}$

➡ $h\, x(t-\tau)$

➡ $y(t) = h\, x(t-\tau) + v(t)$

Assumes TX & RX perfectly synched

Carrier Frequency Offset

$x(t)$ ➡ $x(t)\times e^{-j2\pi f_c t}$ ➡ $h\,x(t-\tau)e^{-j2\pi f_c t}$ ➡ $h\,x(t-\tau)e^{-j2\pi f_c t}\times e^{j2\pi f_c' t}$

TX & RX are not synched ➡ $h\,x(t-\tau)\,e^{-j2\pi \Delta f_c t}$

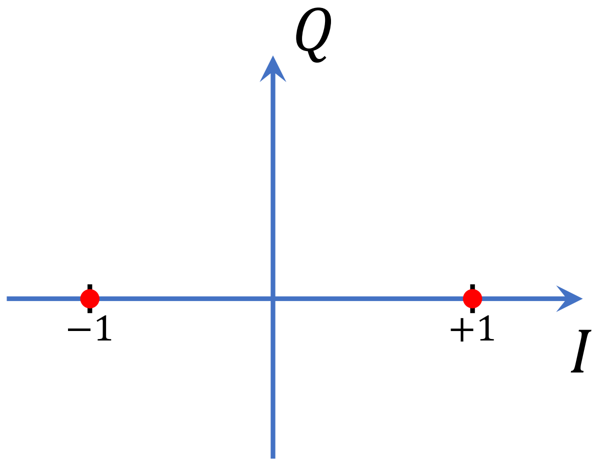
➡ $y(t) = h\,x(t-\tau)e^{-j2\pi \Delta f_c t} + v(t)$

Phase changes with time!

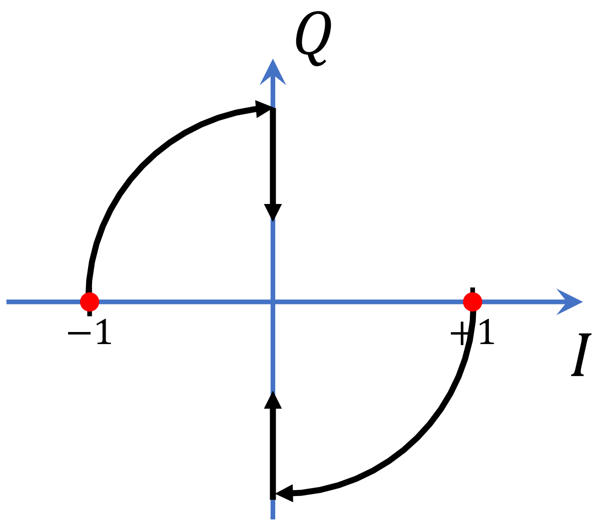
Carrier Frequency Offset

Consider BPSK Modulation.

$$0 \rightarrow -1$$
$$1 \rightarrow +1$$



$x(t)$

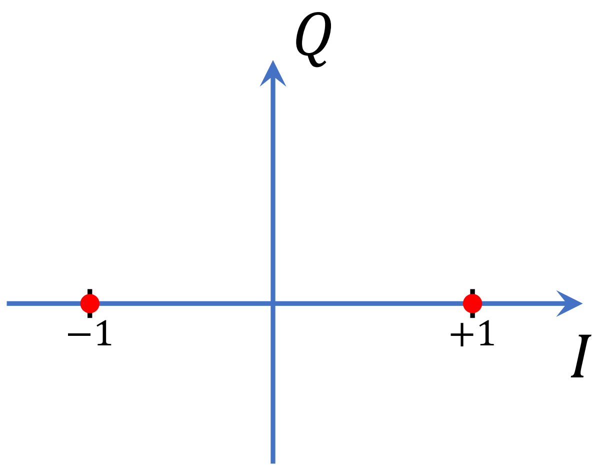


$h\,x(t-\tau)e^{-j2\pi\Delta f_c t} + v(t)$

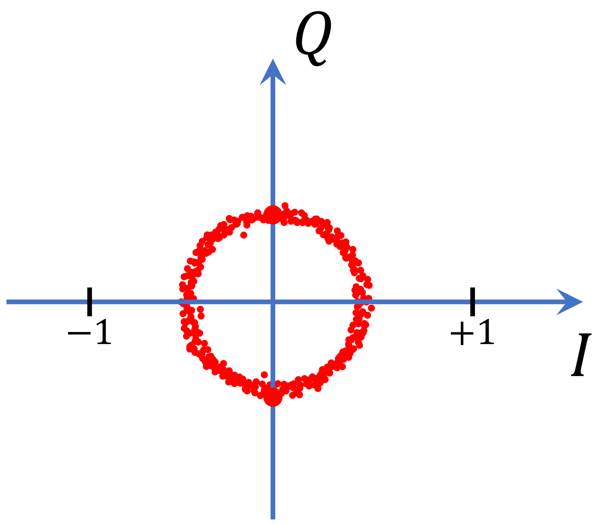
Carrier Frequency Offset

Consider BPSK Modulation.

$$0 \rightarrow -1$$
$$1 \rightarrow +1$$



$x(t)$

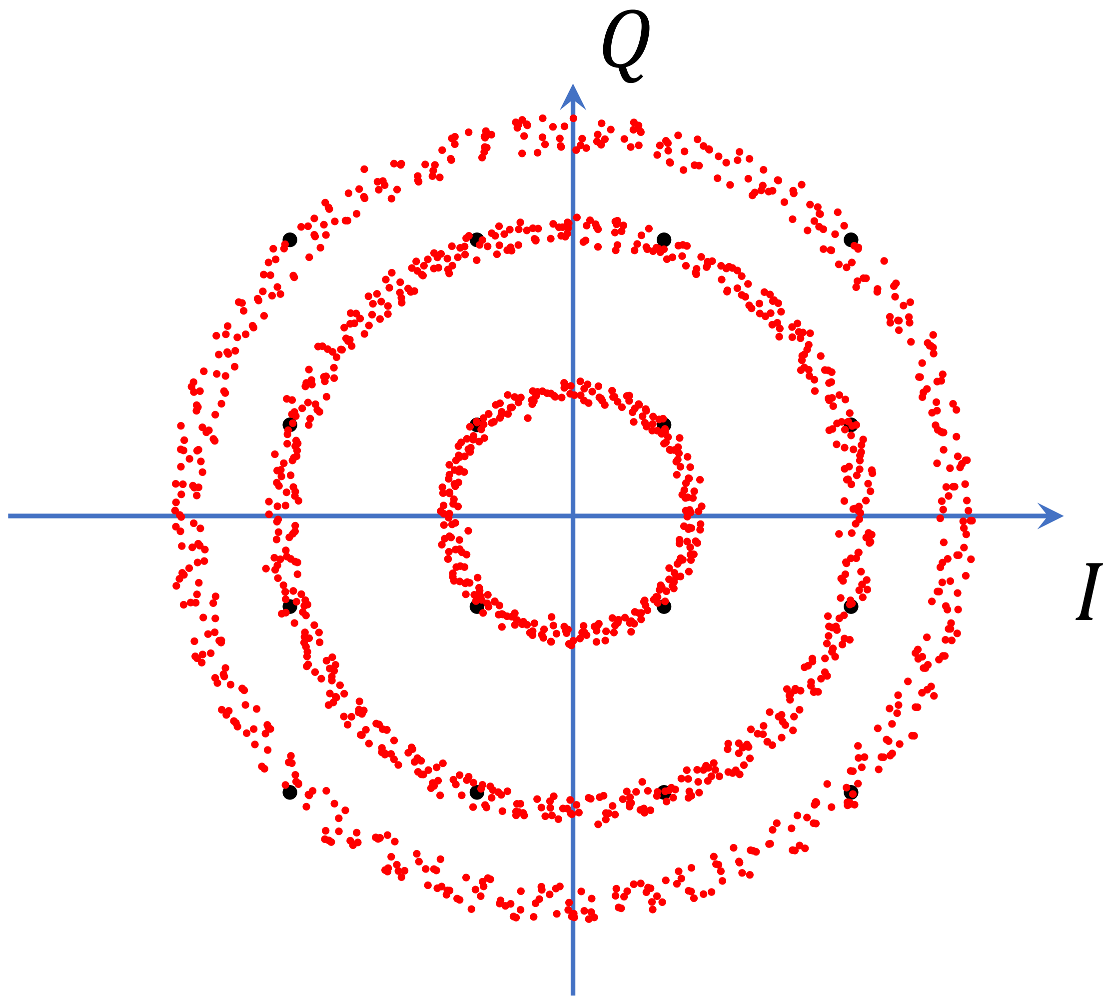


$h\, x(t-\tau)e^{-j2\pi\Delta f_c t} + v(t)$

Impossible to Decode!

Carrier Frequency Offset

Consider 16 QAM Modulation



Need to estimate and correct CFO to decode!

Consequences of CFO

- Cannot decode bits correctly

- Correlation with training sequence does not work for frame synchronization.

Frame Synchronization without CFO

$-1+1-1-1+1$ | $\cdots -1+1-1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d] + v[n]$

$$R[n] = \frac{1}{N}\left|\sum_{k=0}^{N-1} t^*[k]y[n+k]\right|^2$$

$$= \frac{1}{N}\left|\sum_{k=0}^{N-1} t^*[k]h \cdot t[n+k-d] + \sum_{k=0}^{N-1} t^*[k]v[n+k]\right|^2$$

Frame Synchronization without CFO

$-1 + 1 - 1 - 1 + 1$ | $\cdots - 1 + 1 - 1 - 1 + 1 + 1 + 1 - 1 \cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d] + v[n]$

$$R[n] = \frac{1}{N}\left|\sum_{k=0}^{N-1} t^*[k]y[n+k]\right|^2$$

$$= \frac{1}{N}\left|\sum_{k=0}^{N-1} t^*[k]h \cdot t[n+k-d] + \sum_{k=0}^{N-1} t^*[k]v[n+k]\right|^2$$

0

Frame Synchronization without CFO

$-1+1-1-1+1$	$\cdots -1+1-1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d] + v[n]$

$$R[n] = \frac{1}{N}\left|\sum_{k=0}^{N-1} t^*[k]y[n+k]\right|^2$$

$$= \frac{1}{N}\left|\sum_{k=0}^{N-1} t^*[k]h \cdot t[n+k-d]\right|^2 = |h|^2 \begin{cases} 1 & if \quad n=d \\ 1/N & if \quad n \neq d \end{cases}$$

Frame Synchronization with CFO

$-1 + 1 - 1 - 1 + 1$ | $\cdots - 1 + 1 - 1 - 1 + 1 + 1 + 1 - 1 \cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d] e^{-j2\pi\Delta f_c n T_s} + v[n]$

$$R[n] = \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k] \times h \cdot t[n+k] e^{-j2\pi\Delta f_c (n+k) T_s} + \sum_{k=0}^{N-1} t^*[k] v[k] \right|^2$$

Frame Synchronization with CFO

$-1+1-1-1+1$	$\cdots -1+1-1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d]e^{-j2\pi\Delta f_c n T_s} + v[n]$

$$R[n] = \frac{1}{N}\left|\sum_{k=0}^{N-1} t^*[k]\times h \cdot t[n+k]e^{-j2\pi\Delta f_c (n+k)T_s} + \cancelto{0}{\sum_{k=0}^{N-1} t^*[k]v[k]}\right|^2$$

Frame Synchronization with CFO

$-1+1-1-1+1$	$\cdots -1+1-1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d]e^{-j2\pi\Delta f_c n T_s} + v[n]$

$$R[n] = \frac{1}{N}|h|^2 \left|\sum_{k=0}^{N-1} t^*[k]t[n+k-d]e^{-j2\pi\Delta f_c (n+k) T_s}\right|^2$$

$$= \frac{|h|^2}{N}\begin{cases} \left|\sum_{k=0}^{N-1} e^{-j2\pi\Delta f_c (n+k) T_s}\right|^2 & if \quad n = d \\ 1 & if \quad n \neq d \end{cases}$$

Frame Synchronization with CFO

$-1+1-1-1+1$	$\cdots -1+1-1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h\cdot t[n-d]e^{-j2\pi\Delta f_c nT_s} + v[n]$

$$R[n] = \frac{1}{N}|h|^2\left|\sum_{k=0}^{N-1} t^*[k]t[n+k-d]e^{-j2\pi\Delta f_c(n+k)T_s}\right|^2$$

$$= \frac{|h|^2}{N}\begin{cases}\left|\sum_{k=0}^{N-1} e^{-j2\pi\Delta f_c(n+k)T_s}\right|^2 & if \quad n=d \\ 1 & if \quad n\neq d\end{cases}$$

0

$\approx$ noise

Frame Synchronization with CFO

$-1+1-1-1+1$	$-1+1-1-1+1$	$\cdots-1+1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d]e^{-j2\pi\Delta f_c nT_s} + v[n]$
- Repeat training sequence: $y[n+N] = y[n] = t[n]$
- Correlate first N samples with next N samples.

$$R[n] = \left|\sum_{k=0}^{N-1} y^*[n+k]y[n+k+N]\right|$$

$$if\ n = d, \quad = \left|\sum_{k=0}^{N-1} h^*t^*[k]e^{j2\pi\Delta f_c(n+k)T_s} \cdot ht[k]e^{-j2\pi\Delta f_c(n+N+k)T_s}\right|$$

$$= \left||h|^2 e^{-j2\pi\Delta f_c NT_s} \sum_{k=0}^{N-1} t^*[k]t[k]\right| = N|h|^2$$

Frame Synchronization with CFO

$-1+1-1-1+1$	$-1+1-1-1+1$	$\cdots -1+1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d]e^{-j2\pi\Delta f_c n T_s} + v[n]$
- Repeat training sequence: $y[n+N] = y[n] = t[n]$
- Correlate first N samples with next N samples.

$$R[n] = \left|\sum_{k=0}^{N-1} y^*[n+k]y[n+k+N]\right|$$

$$\left.\begin{array}{l} if\ n = d,\ \ R[n] = N|h|^2 \\ \\ if\ n \neq d,\ \ R[n] < R[d] \end{array}\right\} \text{Find } n \text{ that maximizes } R[n]$$

Frame Synchronization with CFO

$-1+1-1-1+1$	$-1+1-1-1+1$	$\cdots-1+1-1+1+1+1-1\cdots$

- $t[n]$ is training sequence of length N
- Discrete samples: $y[n] = h \cdot t[n-d]e^{-j2\pi\Delta f_c n T_s} + v[n]$
- Repeat training sequence: $y[n+N] = y[n] = t[n]$
- Correlate first N samples with next N samples.

$$R[n] = \frac{\left|\sum_{k=0}^{N-1} y^*[n+k]y[n+k+N]\right|}{\sqrt{\sum_{k=0}^{N-1}|y[n+k]|^2}\sqrt{\sum_{k=0}^{N-1}|y[n+k+N]|^2}} \begin{cases} = 1 & if \quad n = d \\ < 1 & if \quad n \neq d \end{cases}$$

Consequences of CFO

- Cannot decode bits correctly

  Estimate & Correct for CFO

- ✓ Correlation with training sequence does not work for frame synchronization.

  Repeat training sequence & correlate consecutive sequences of received samples.

Estimating & Correcting for CFO

$-1+1-1-1+1$	$-1+1-1-1+1$	$\cdots-1+1-1+1+1+1-1\cdots$

- Use training sequence $t[n]$ of length N repeated twice

- Discrete samples: $y[n] = h \cdot t[n-d]e^{-j2\pi\Delta f_c nT_s} + v[n]$

$$A = \sum_{k=0}^{N-1} y^*[d+k]y[d+k+N]$$

$$= \sum_{k=0}^{N-1} h^*t^*[k]e^{j2\pi\Delta f_c(d+k)T_s} \cdot ht[k]e^{-j2\pi\Delta f_c(d+N+k)T_s} = N|h|^2e^{-j2\pi\Delta f_c NT_s}$$

Estimate CFO: $\Delta f_c = \frac{\angle A}{2\pi NT_s}$

Correct CFO: $y[n]\times e^{j2\pi\Delta f_c nT_s}$

Estimating & Correcting for CFO

Estimate CFO: $\Delta f_c = \frac{\angle A}{2\pi N T_s}$

Correct CFO: $y[n] \times e^{j2\pi\Delta f_c n T_s}$

Phase Wraps Around 2π

$$-\pi \le \angle A \le \pi$$

Estimating & Correcting for CFO

Estimate CFO: $\Delta f_c = \dfrac{\angle A}{2\pi N T_s}$

Correct CFO: $y[n] \times e^{j2\pi\Delta f_c n T_s}$

Phase Wraps Around 2π

$$|\angle A| \leq \pi$$

$$\Delta f_c \leq \frac{1}{2NT_s}$$

- N must not be too large to correctly estimate CFO

e.g., $f_c = 5$ GHz, Clock Precision: 20 ppm, Bandwidth $= 10$ MHz

$$\Delta f_c = 100 \text{ KHz}, T_s = 0.1\ \mu s$$

$$N \leq 50$$

Estimating & Correcting for CFO

Estimate CFO: $\Delta f_c = \frac{\angle A}{2\pi N T_s}$

Correct CFO: $y[n] \times e^{j2\pi \Delta f_c n T_s}$

Phase Wraps Around 2π

$$|\angle A| \leq \pi$$

$$\Downarrow$$

$$\Delta f_c \leq \frac{1}{2NT_s}$$

- N must not be too large to correctly estimate CFO
- N must be large enough to average out the noise

Estimating & Correcting for CFO

Estimate CFO: $\Delta f_c = \dfrac{\angle A}{2\pi N T_s}$

Phase Wraps Around 2π

$$|\angle A| \leq \pi$$

$$\Delta f_c \leq \frac{1}{2NT_s}$$

- N must not be too large to correctly estimate CFO
- N must be large enough to average out the noise

Correct CFO: $y[n] \times e^{j2\pi\Delta f_c n T_s}$

How do we know exact index value of n?

$$y[n] \times e^{j2\pi\Delta f_c (n+k) T_s}$$

$$= hx[n]e^{-j2\pi\Delta f_c n T_s} \times e^{j2\pi\Delta f_c (n+k) T_s}$$

$$= hx[n]e^{j2\pi\Delta f_c k T_s}$$

$$= \left(he^{j2\pi\Delta f_c k T_s}\right)x[n]$$

$$= h'x[n]$$

- It does not matter!
- Lumped with Channel Equalization

Consequences of CFO

- ✓ Cannot decode bits correctly

  Estimate & Correct for CFO

- ✓ Correlation with training sequence does not work for frame synchronization.

  Repeat training sequence & correlate consecutive sequences of received samples.

Estimate & Correct for CFO

So Far: Pilot Assisted Carrier Acquisition

- **Residual CFO:**

$$y(t) = h\, x(t - \tau) e^{-j2\pi \Delta f_c t} + v(t)$$

$$\Delta f_c = d f_c + \delta f_c$$

Coarse CFO   Residual CFO

We estimated and corrected for coarse CFO!

Even small residual can accumulate over time to create large phase: $e^{-j2\pi \delta f_c t}$

Need to track the phase

Estimate & Correct for CFO

So Far: Pilot Assisted Carrier Acquisition

- **Residual CFO:** δf_c

- **Initial Carrier Phase:**

$$y(t) = h\,x(t-\tau)e^{-j(2\pi f_c t+\phi)} \times e^{j(2\pi f_c' t+\theta)}$$

$$y(t) = h\,x(t-\tau)e^{-j(2\pi \Delta f_c t+\phi-\theta)}$$

In principle, not a problem!

Lump it with Channel Equalization.

However, phase might not be stable over time.

Need to track the phase

Estimate & Correct for CFO

So Far: Pilot Assisted Carrier Acquisition

- **Residual CFO:** δf_c

- **Initial Carrier Phase:** ϕ

Phase Tracking

Extract the phase of the carrier and track it over time.

Phase Tracking

Many Methods:

- Squared Difference Loop
- Phased Locked Loop
- Costas Loop
- …

Squared Difference Loop Method

Consider simple BPSK: $r(t) = \pm 1\cos(2\pi f_0 t + \phi)$

- Square the signal: $r^2(\mathrm{t}) = \cos^2(2\pi f_0 t + \phi)$
- Bandpass filter it at $2f_0$: $r_p(t) = \cos(4\pi f_0 t + 2\phi)$
- Sample it: $r_p(kT_s) = \cos(4\pi f_0 kT_s + 2\phi)$

Goal is to find & track ϕ

Find θ that minimizes the average squared error:

$$J_{SD}(\theta) = \frac{1}{4}\mathrm{LPF}\left\{\left(r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta)\right)^2\right\}$$

$$= \frac{1}{4}(1 - \cos(2\phi - 2\theta))$$

Squared Difference Loop Method

Find θ that minimizes the average squared error:

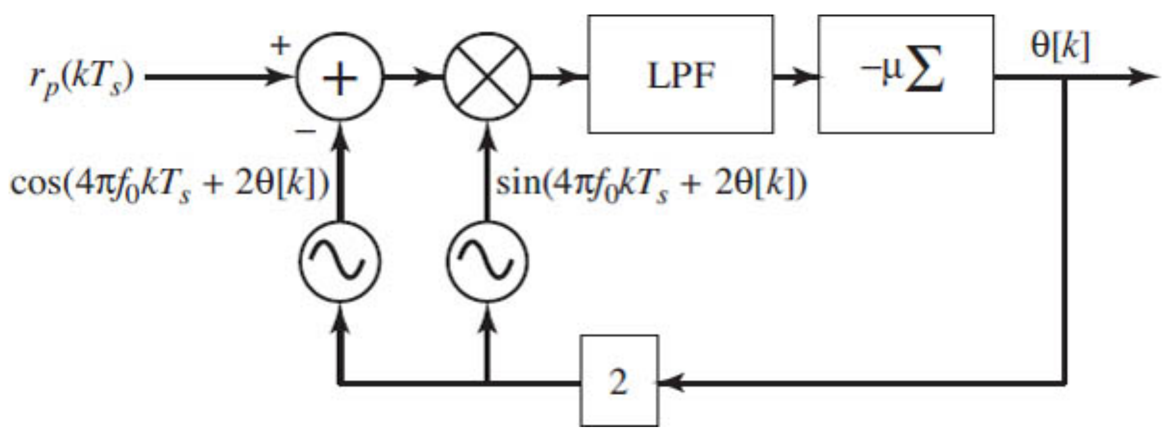
$$J_{SD}(\theta) = \frac{1}{4}\mathrm{LPF}\left\{\left(r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta)\right)^2\right\}$$

Minimize using Gradient Descent:

$$\theta[k+1] = \theta[k] - \mu \left.\frac{dJ_{SD}(\theta)}{d\theta}\right|_{\theta=\theta[k]}$$

$$\theta[k+1] = \theta[k] - \mu \mathrm{LPF}\left\{\left(r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta[k])\right)\sin(4\pi f_0 kT_s + 2\theta[k])\right\}$$

$$\theta[k+1] = \theta[k] - \mu \sin(2\phi - 2\theta[k])$$



Phased Lock Loop Method

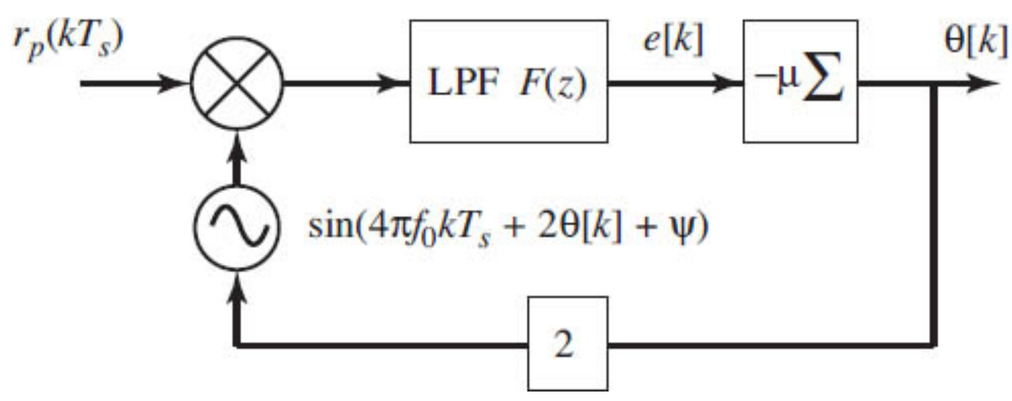
Find θ that maximizes correlation:

$$J_{PLL}(\theta) = \frac{1}{2}\text{LPF}\{r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta)\}$$

Minimize using Gradient Descent:

$$\theta[k+1] = \theta[k] - \mu \left.\frac{dJ_{PLL}(\theta)}{d\theta}\right|_{\theta=\theta[k]}$$

$$\theta[k+1] = \theta[k] - \mu\text{LPF}\{\left(r_p(kT_s)\sin(4\pi f_0 kT_s + 2\theta[k])\right)\}$$



Costas Loop Method

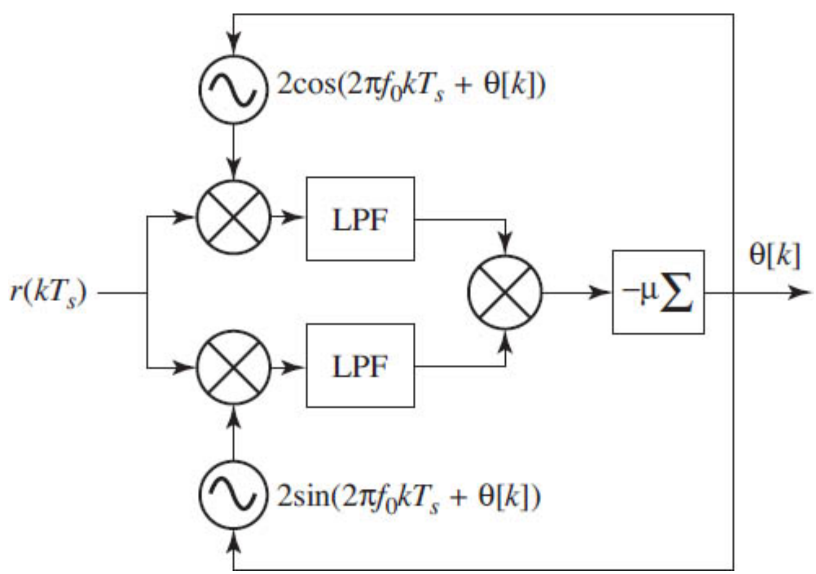
Find θ that maximizes correlation but operates directly on received signal:

$$J_C(\theta) = \frac{1}{2}\mathrm{LPF}\{(\mathrm{LPF}\{r(kT_s)\cos(2\pi f_0 kT_s + \theta)\})\}^2$$

Minimize using Gradient Descent:

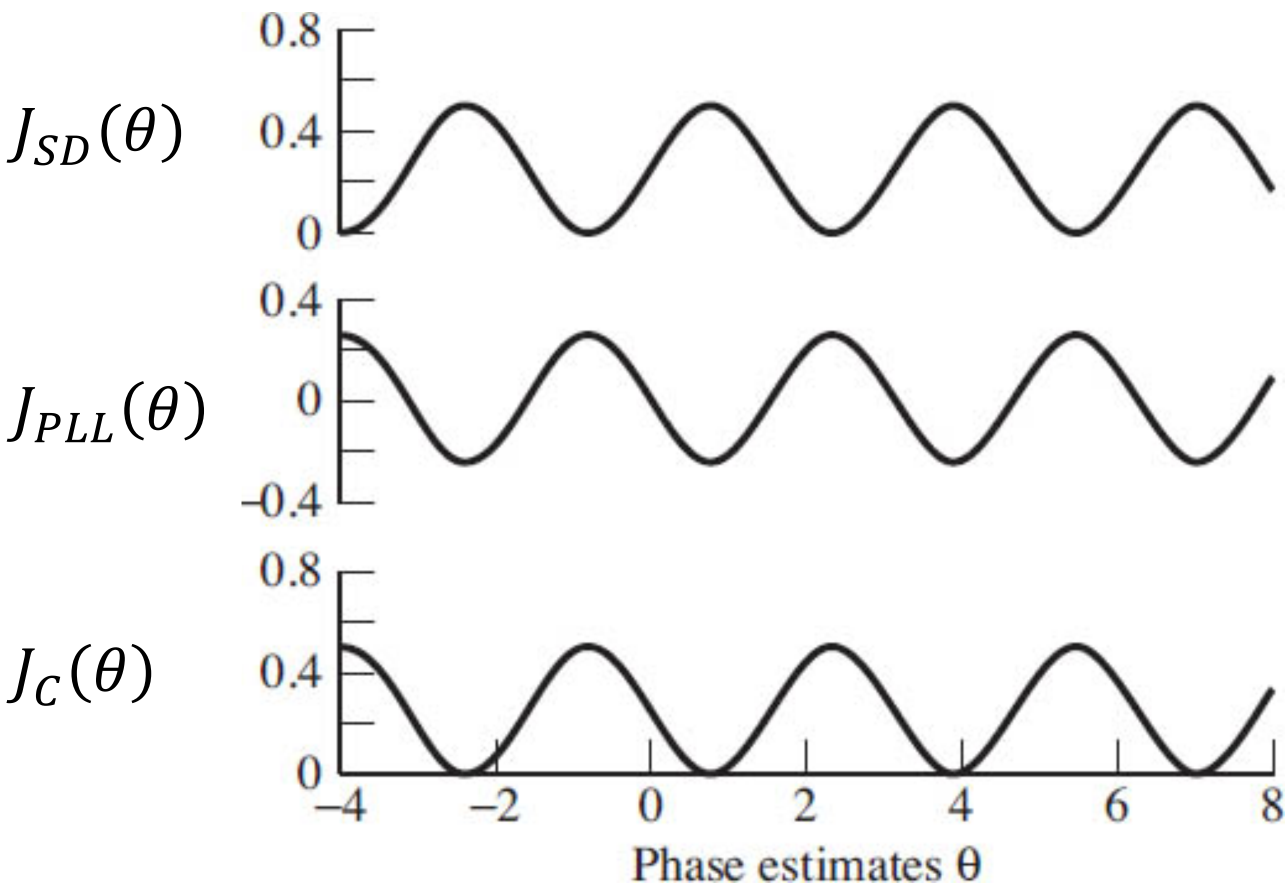
$$\theta[k+1] = \theta[k] - \mu \left.\frac{dJ_c(\theta)}{d\theta}\right|_{\theta=\theta[k]}$$

$$\theta[k+1] = \theta[k] - \mu\mathrm{LPF}\{(r(kT_s)\sin(2\pi f_0 kT_s + \theta[k]))\} \times\mathrm{LPF}\{(r(kT_s)\sin(2\pi f_0 kT_s + \theta[k]))\}$$



Phase Tracking

Similar objective functions



Definitions & Variables

- $x(t)$: Transmitted Signal
- $v(t)$: Additive Gaussian Noise
- $y(t)$: Received Signal
- τ: Time delay of the signal
- h: Single Tap Channel Coefficient.
- α: Attenuation of the channel
- f_c: Carrier Frequency of Transmitter
- $f_c{}'$: Carrier Frequency of Receiver
- Δf_c: Carrier Frequency Offset (CFO)
- df_c: Carrier Frequency Offset
- δf_c: Carrier Frequency Offset
- ϕ: Initial TX Carrier Phase
- θ: Initial RX Carrier Phase
- $(\quad)^*$: Complex Conjugate
- T_s: Symbol time
- $y[n]$: Sampled received signal
- $t[n]$: Training Sequence
- N: Length of the Training Sequence
- $R[n]$: Cross Correlation Function
- d: Signal delay in number of samples
- n: Symbol index
- $r(t)$: Modulated Signal
- f_0: Intermediate Carrier Frequency
- $r_p(t)$: Squared and bandpass filtered $r(t)$
- $J_{SD}(\theta)$: Squared Difference Loop Optimization Function
- $J_{PLL}(\theta)$: Phase Lock Loop Optimization Function
- $J_C(\theta)$: Costas Loop Optimization Function
- μ: Gradient Descent update parameter