ECE 463: Digital Communications Lab.

Lecture 3: Pulse Shaping and Matched Filters

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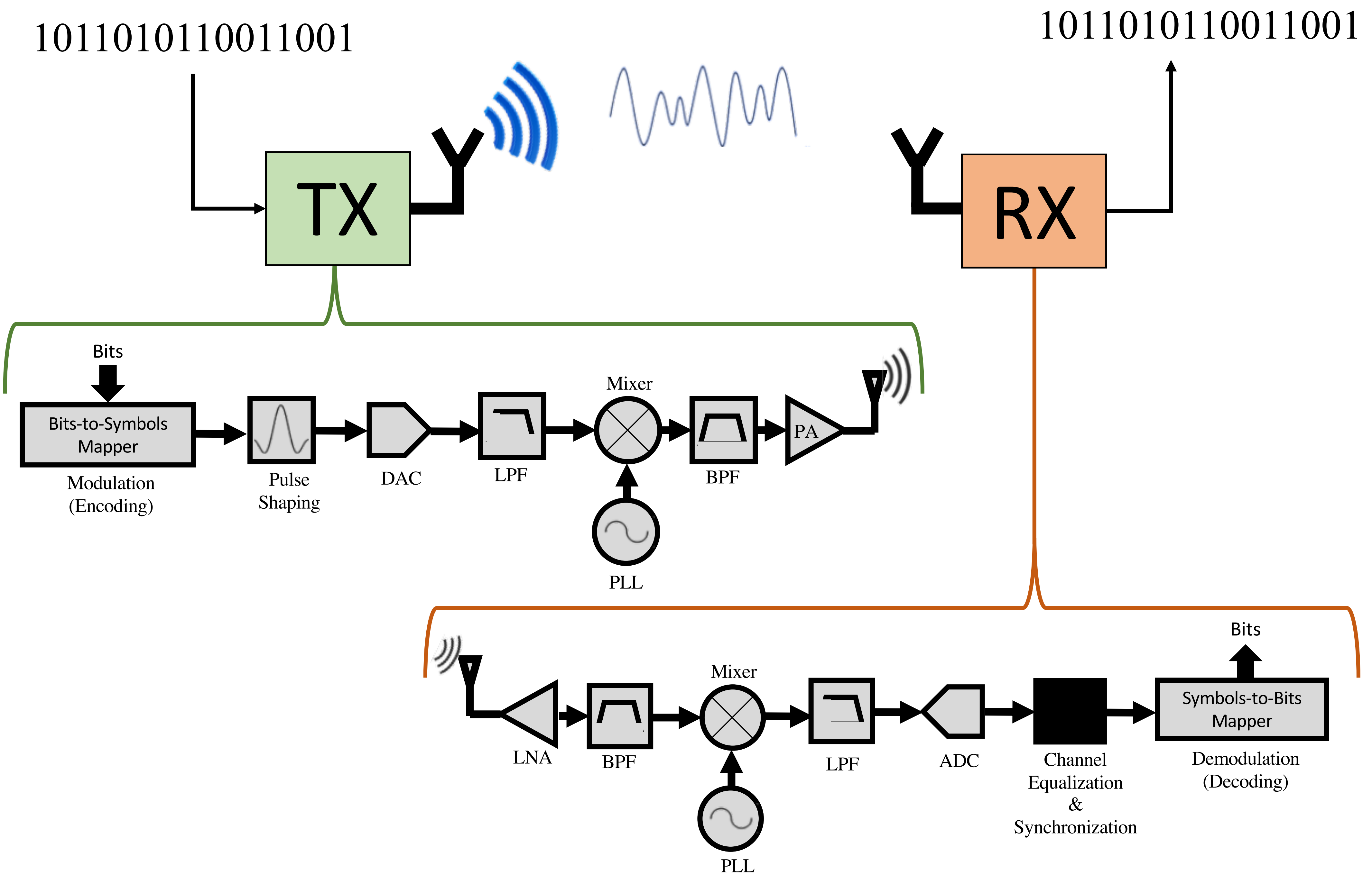
Previous Lecture:

- ✓ Up Conversion & Down Conversion
- ✓ PAM vs QAM Spectral Efficiency
- ✓ Software Defined Radios

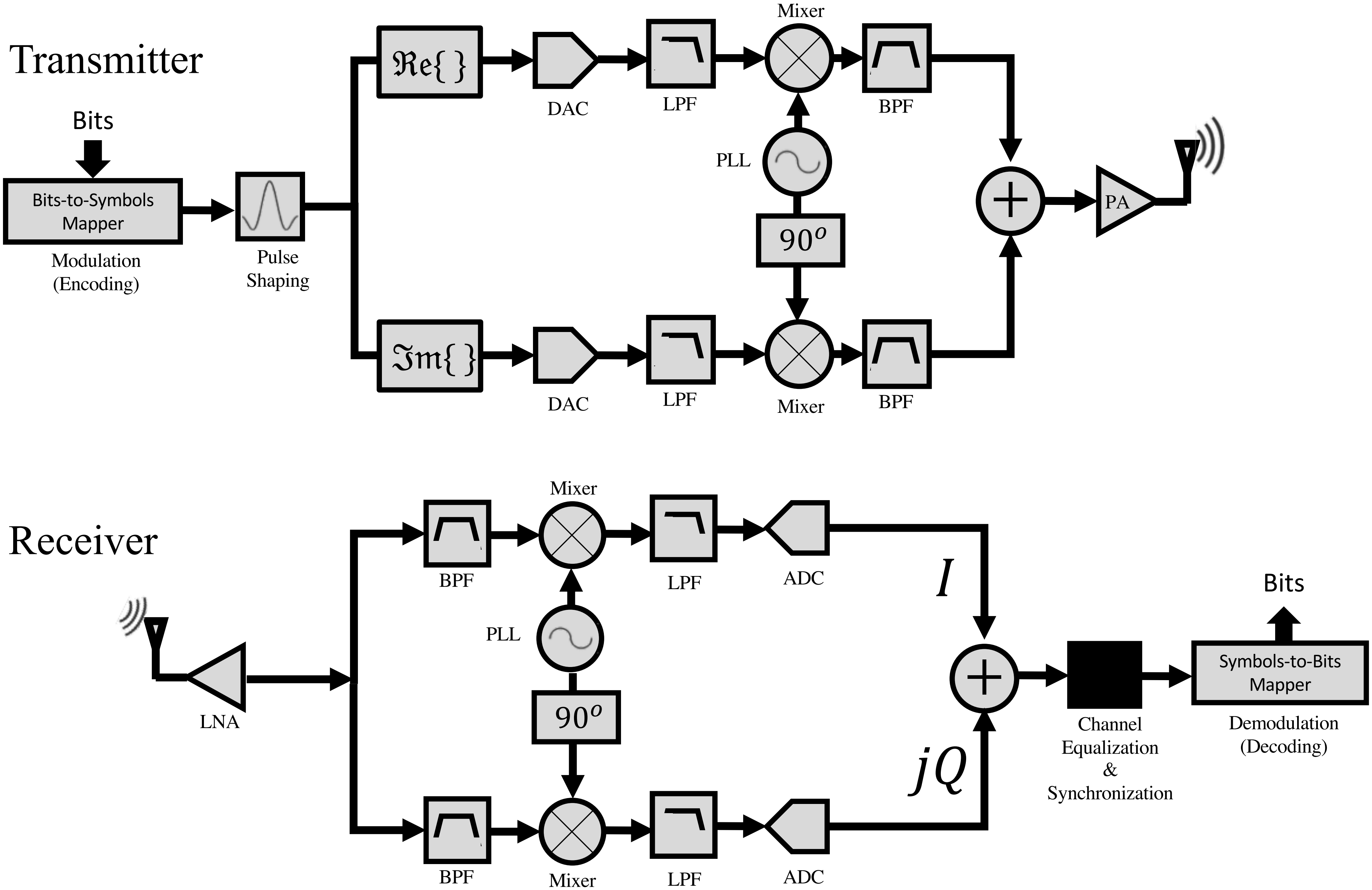
This Lecture:

- ❑ Pulse Shaping Filters
- ❑ Matched Filters
- ❑ Symbol Timing Recovery
- ❑ Eye Diagrams

Digital Communication System



Digital Communication System



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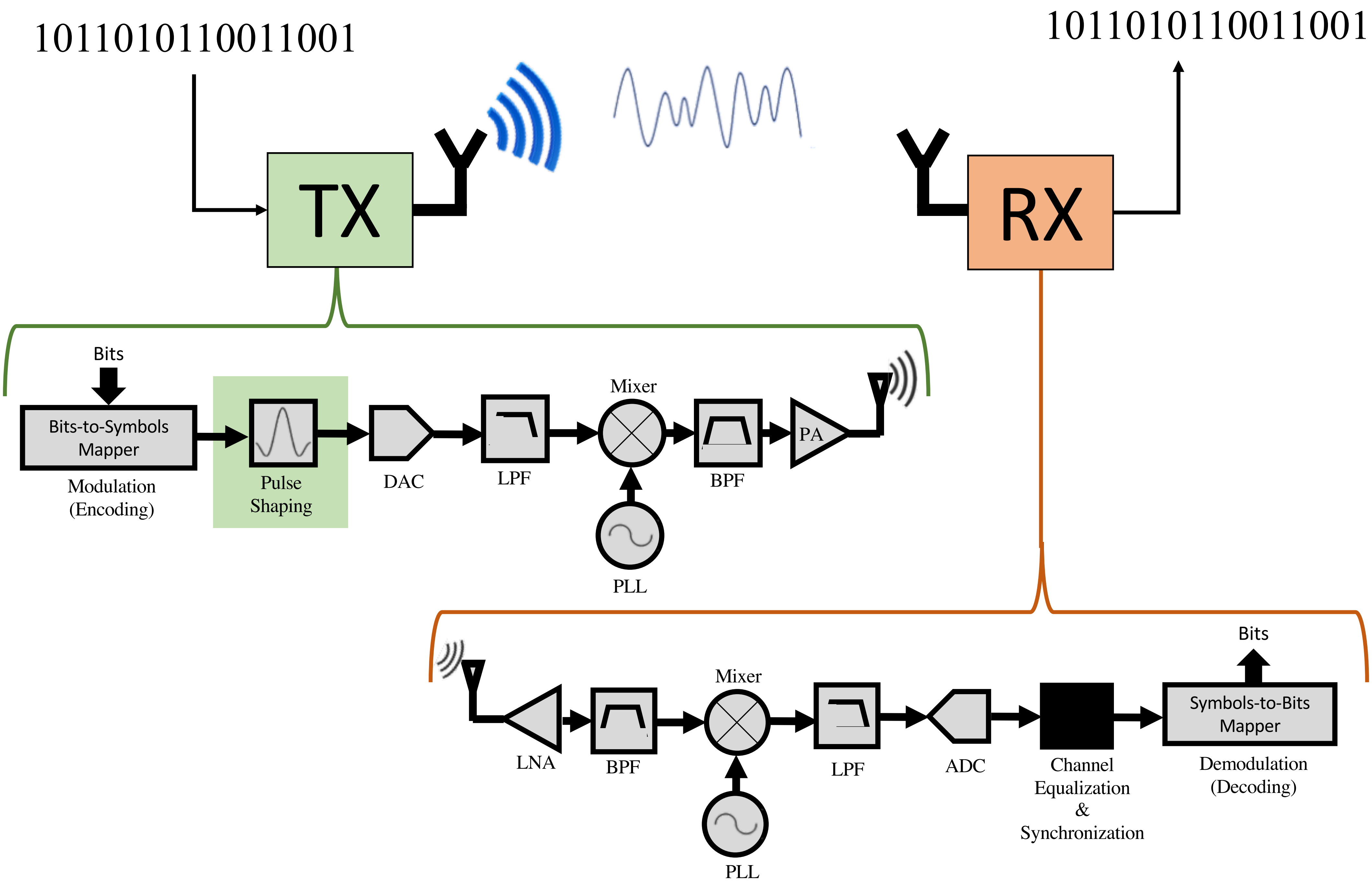

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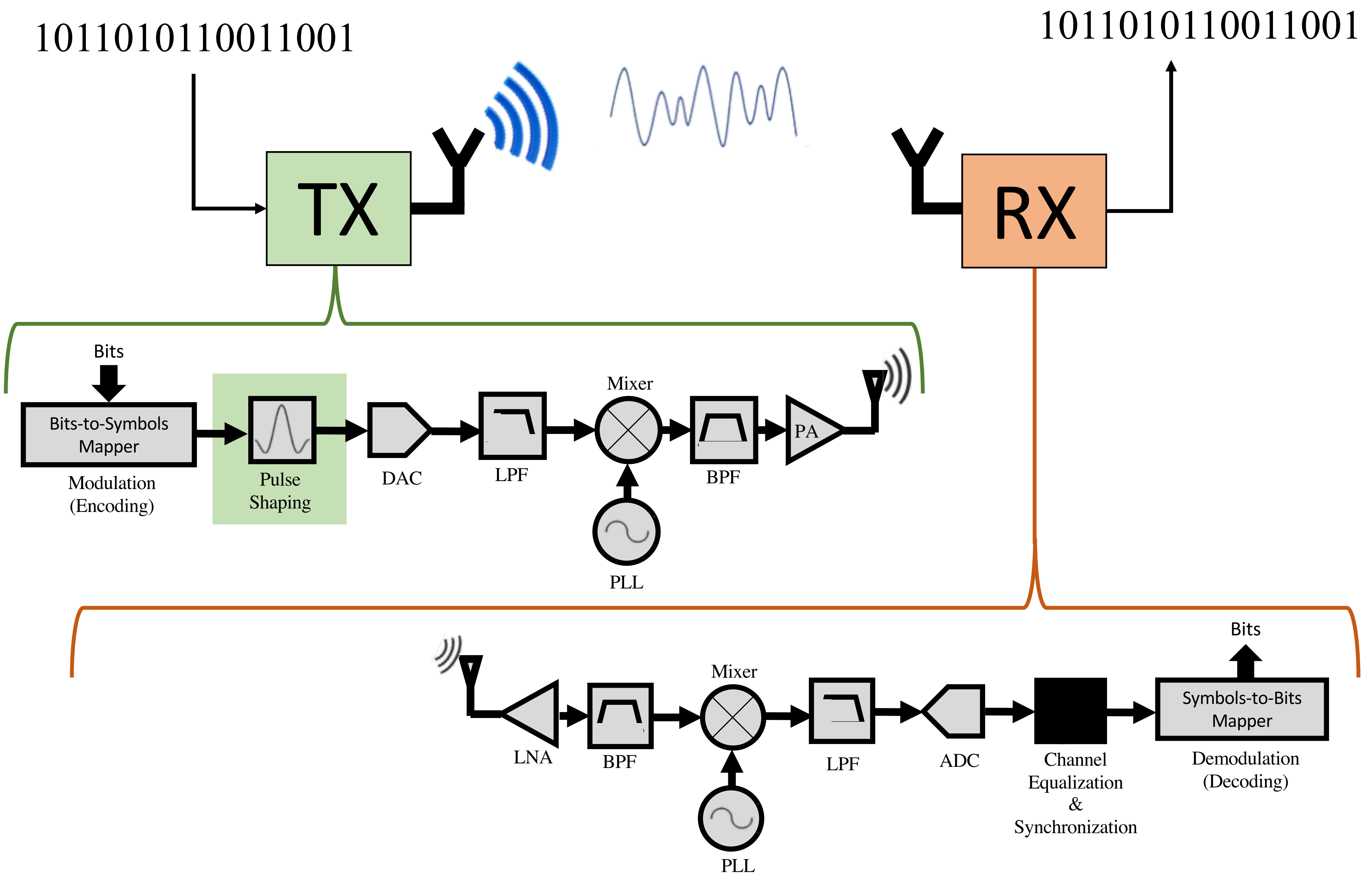

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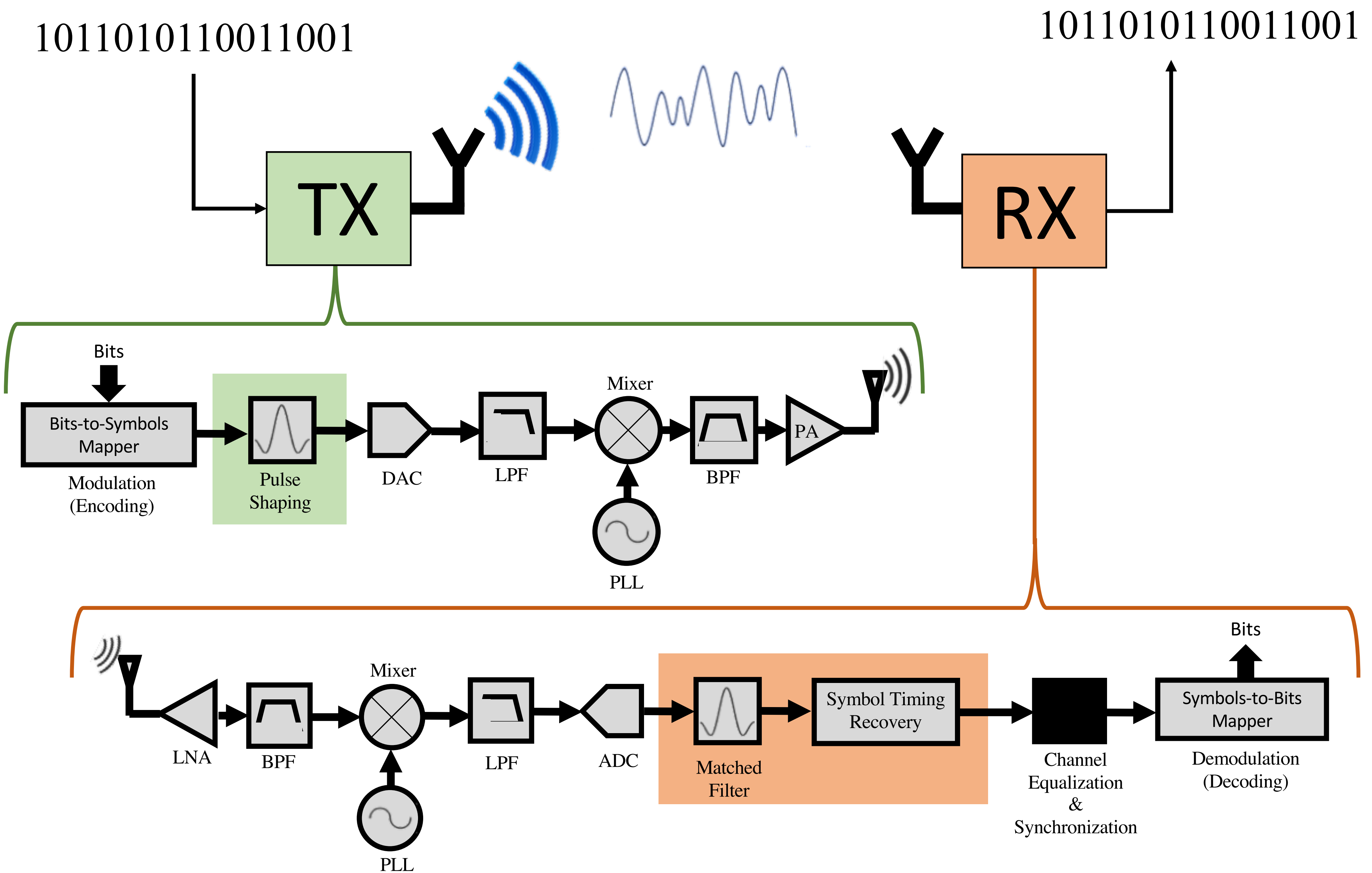
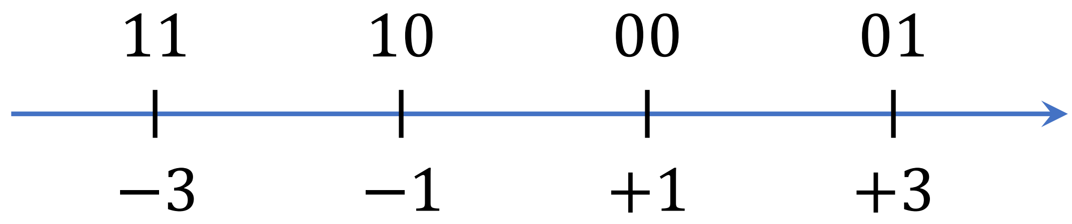
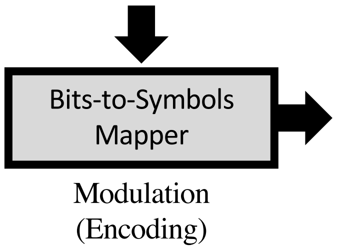


PAM: Pulse Amplitude Modulation



$b[n]$: 1011000110011001

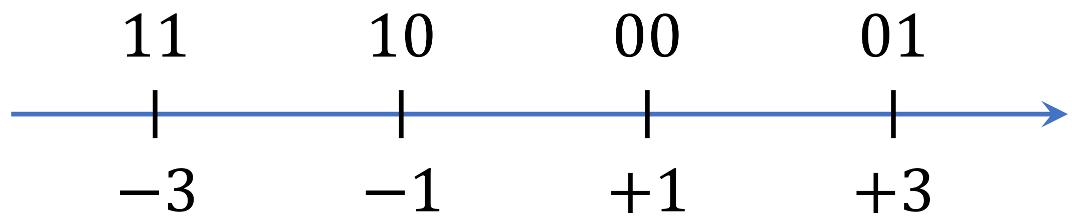


$s[n]$: -1,-3,+1,+3,-1,+1,-1

$$x(t) = \sum_{n=-\infty}^{+\infty} s[n]p(t - nT_b)$$

- T_b: Symbol Time/ Baud Time
- $R_b = \frac{1}{T_b}$: Symbol Rate or Baud Rate
- $p(t)$: pulse shape

Pulse Shaping



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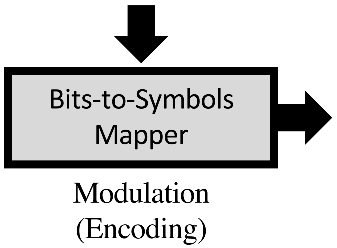

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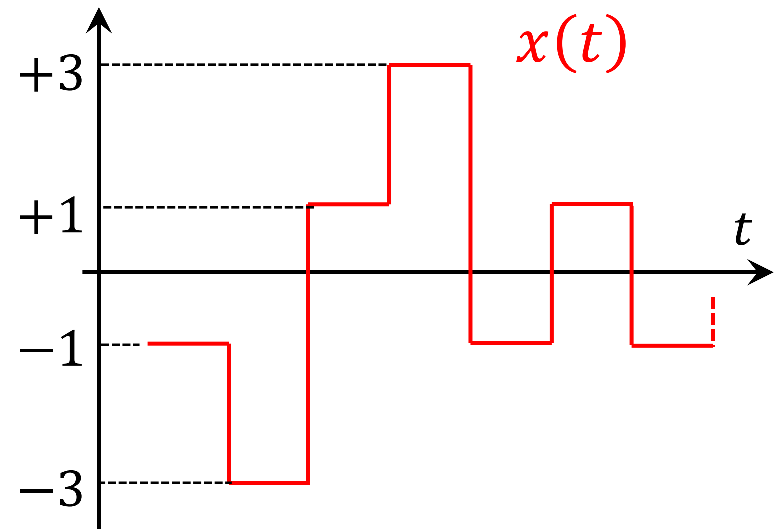
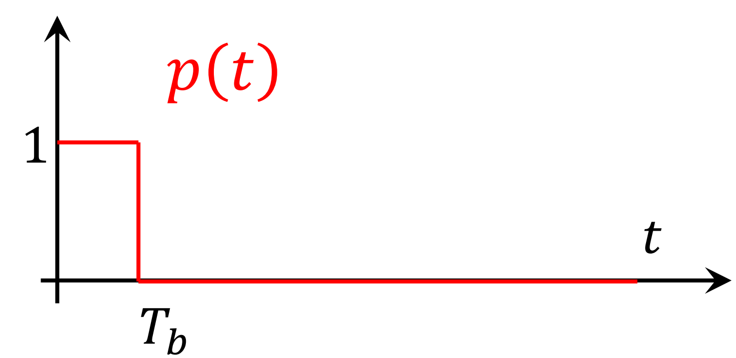


$$x(t) = \sum_{n=-\infty}^{+\infty} s[n]p(t - nT_b)$$

- Simplest pulse shape: Rectangle

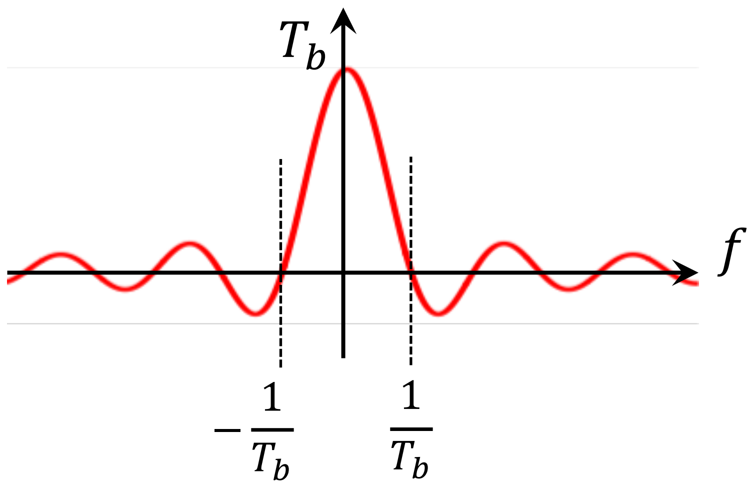
$$p(t) = \Pi\left(\frac{t}{T_b}\right)$$



Pulse Shaping: Rectangular Pulse

$$p(t) = \Pi\left(\frac{t}{T_b}\right)$$

$$P(f) = T_b \operatorname{sinc}(\pi T_b f)$$



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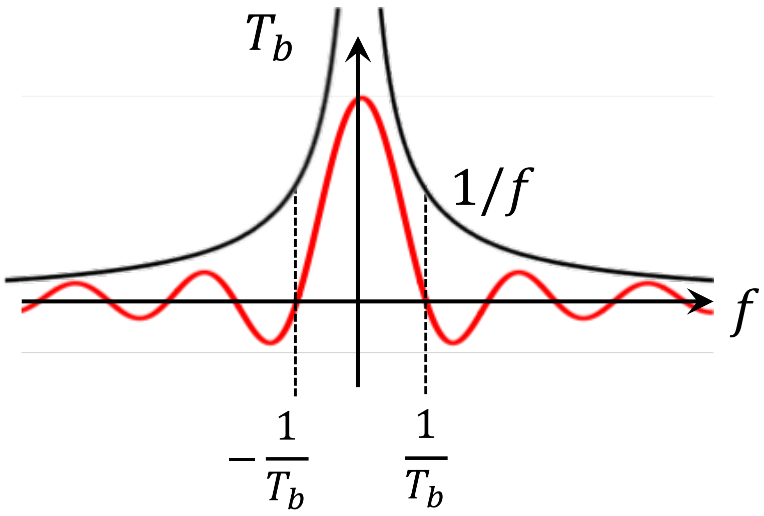

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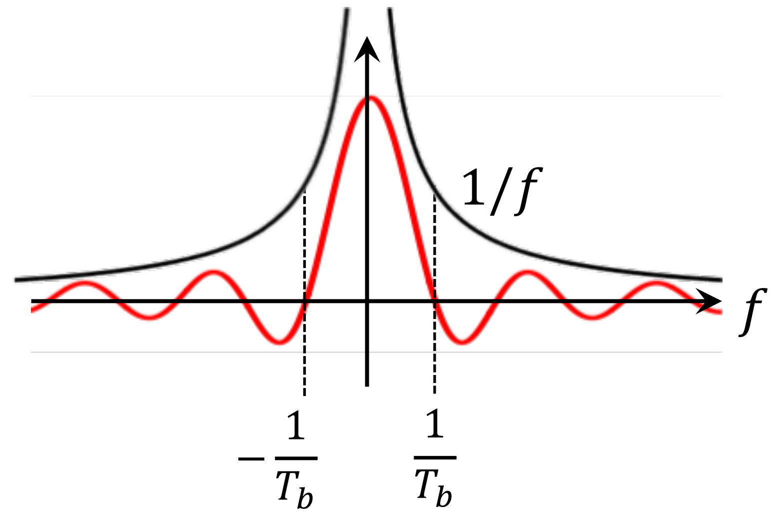
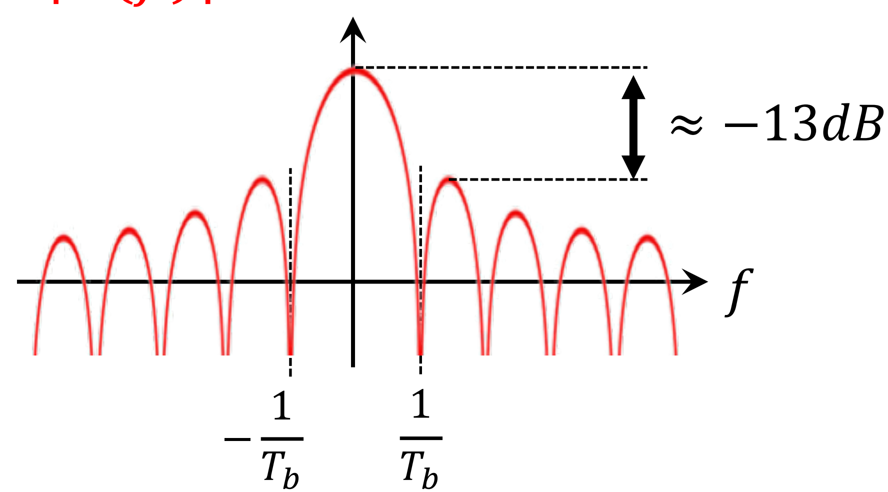


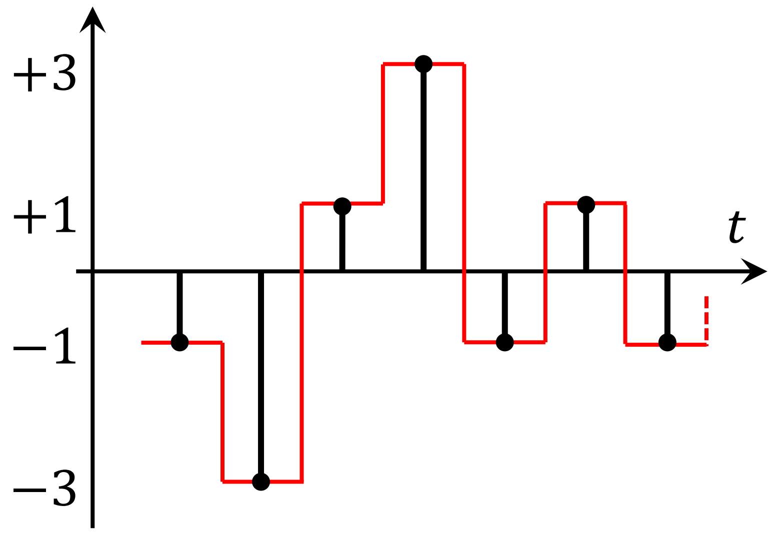
$|P(f)|^2$



- Very wide bandwidth!
- Low spectral efficiency

Pulse Shaping: Rectangular Pulse

Satisfies the Nyquist Criterion For Inter-Symbol-Interference



Sampled Values: $r[n] = -1, -3, +1, +3, -1, +1, -1$ ...

➡ $r[n] = s[n]$ ➡ **No Inter-Symbol-Interference (ISI)**

ISI: $r[n] = s[n] + p(T_b)s[n-1] + p(2T_b)s[n-2] +$ ...

Rectangular Pulse ➡ **No ISI**

Pulse Shaping: Rectangular Pulse

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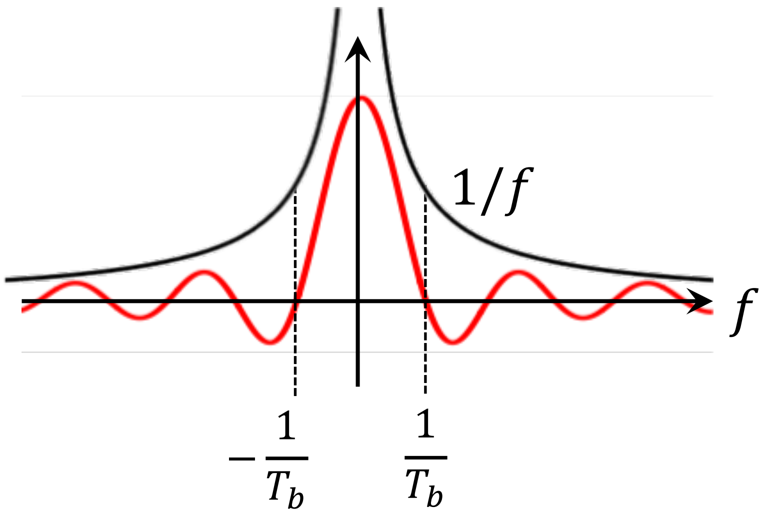

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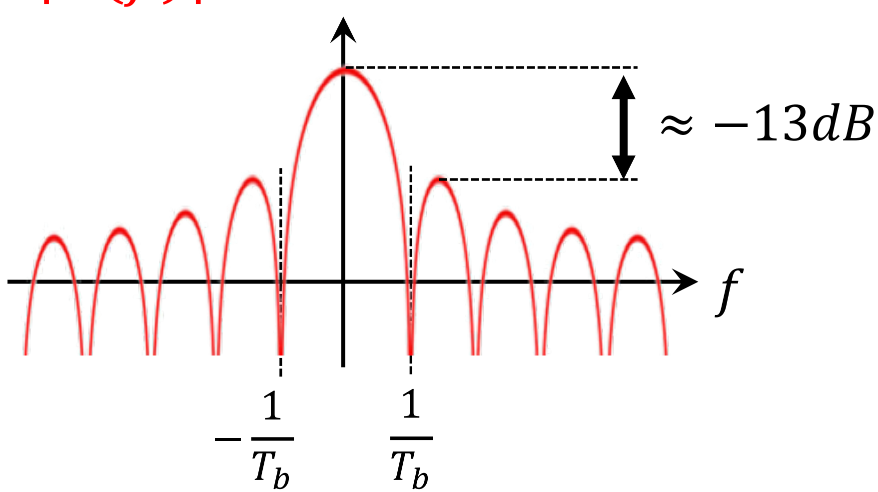
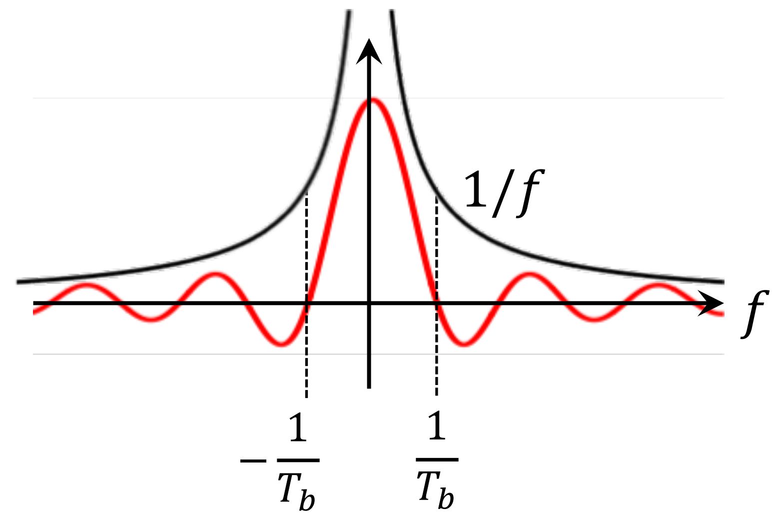


- − Very wide bandwidth!
- − Low spectral efficiency
- + No Inter-Symbol-Interference

Pulse Shaping: Rectangular Pulse

$$p(t) = \Pi\left(\frac{t}{T_b}\right)$$

$$P(f) = T_b \,\mathrm{sinc}(\pi T_b f)$$



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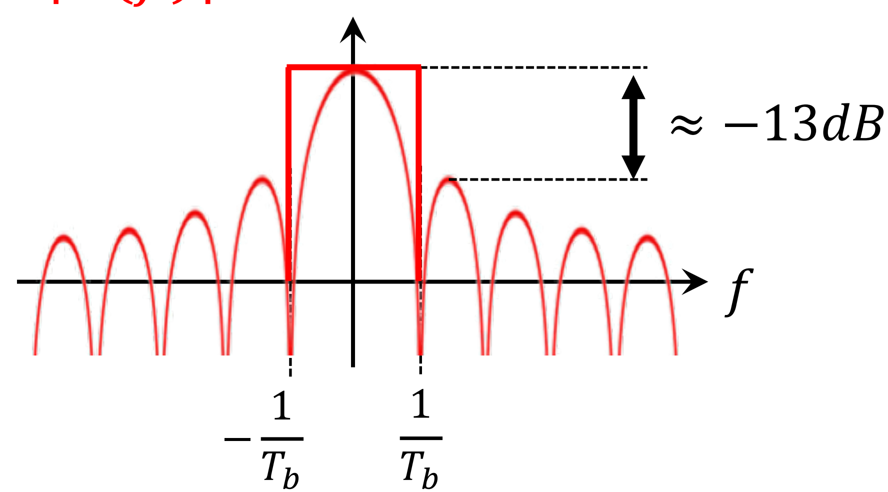

− Very wide bandwidth!

− Low spectral efficiency

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Pulse Shaping: Rectangular Pulse

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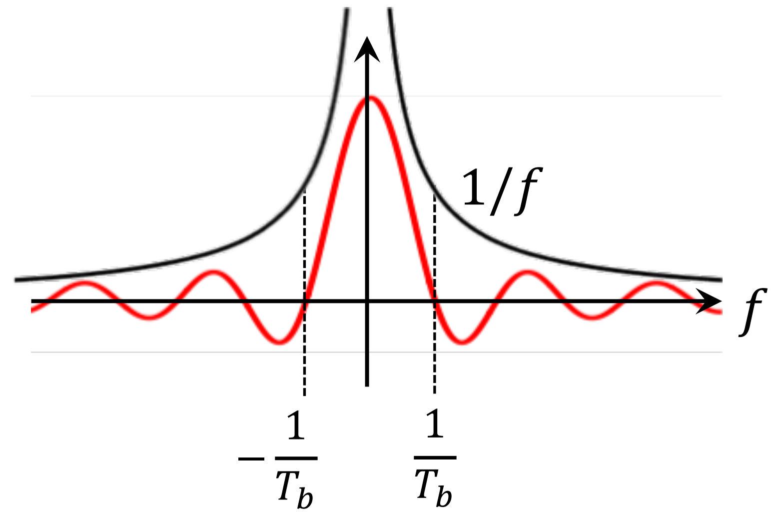
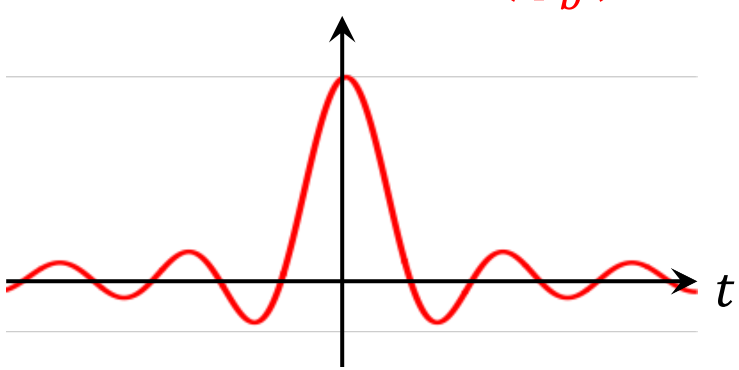


$$p(t) \propto \operatorname{sinc}\left(\frac{2\pi t}{T_b}\right)$$



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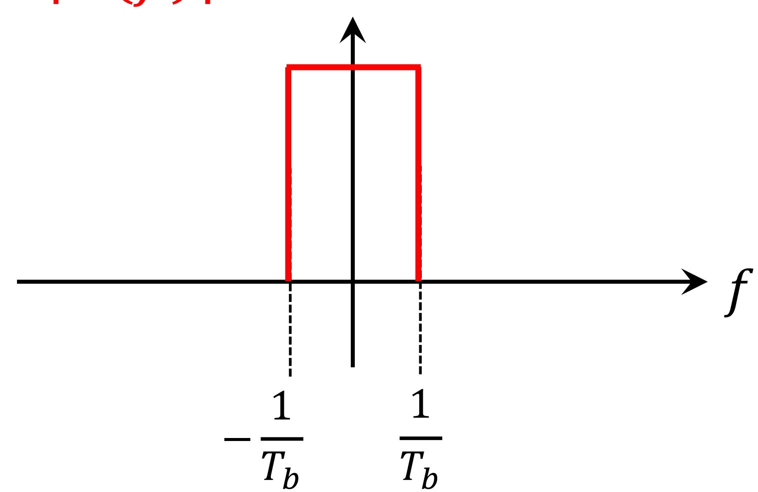
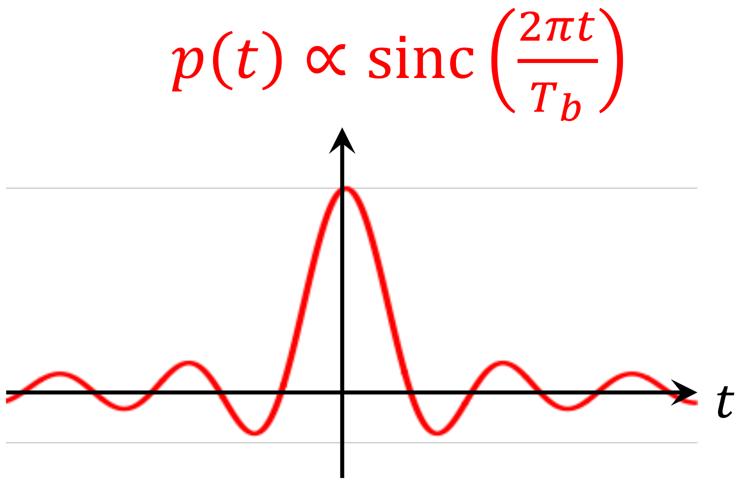
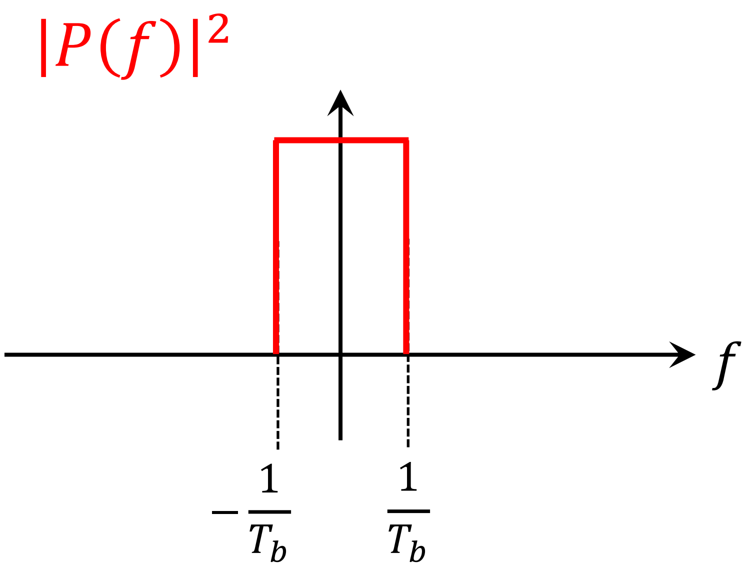


Pulse Shaping: Rectangular Pulse





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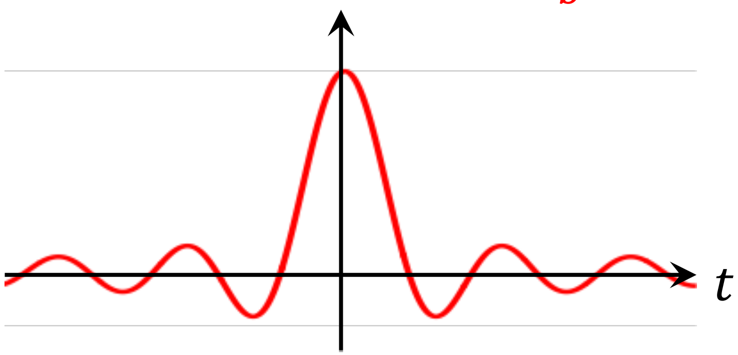

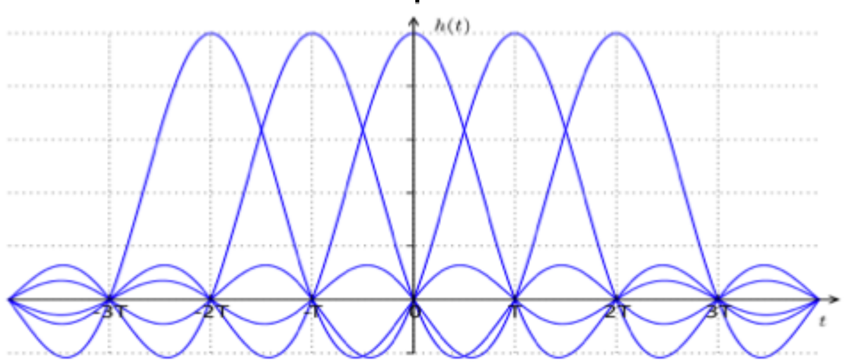

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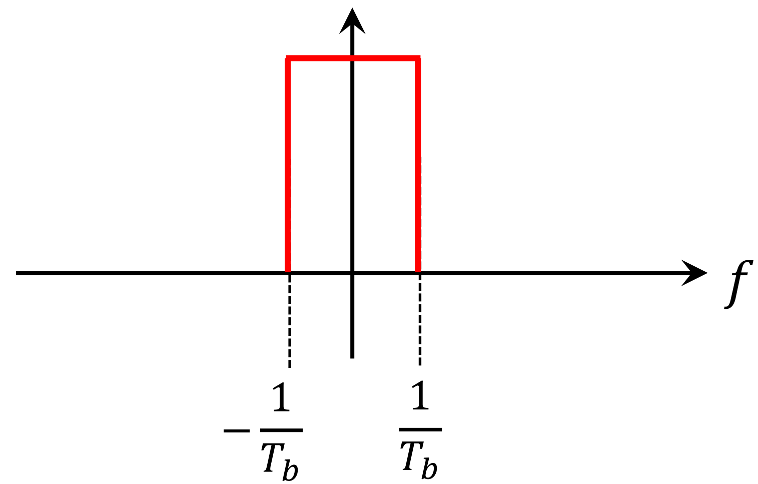


− Infinite Response → Impossible to realize in practice

− High ISI if sampling is not perfect

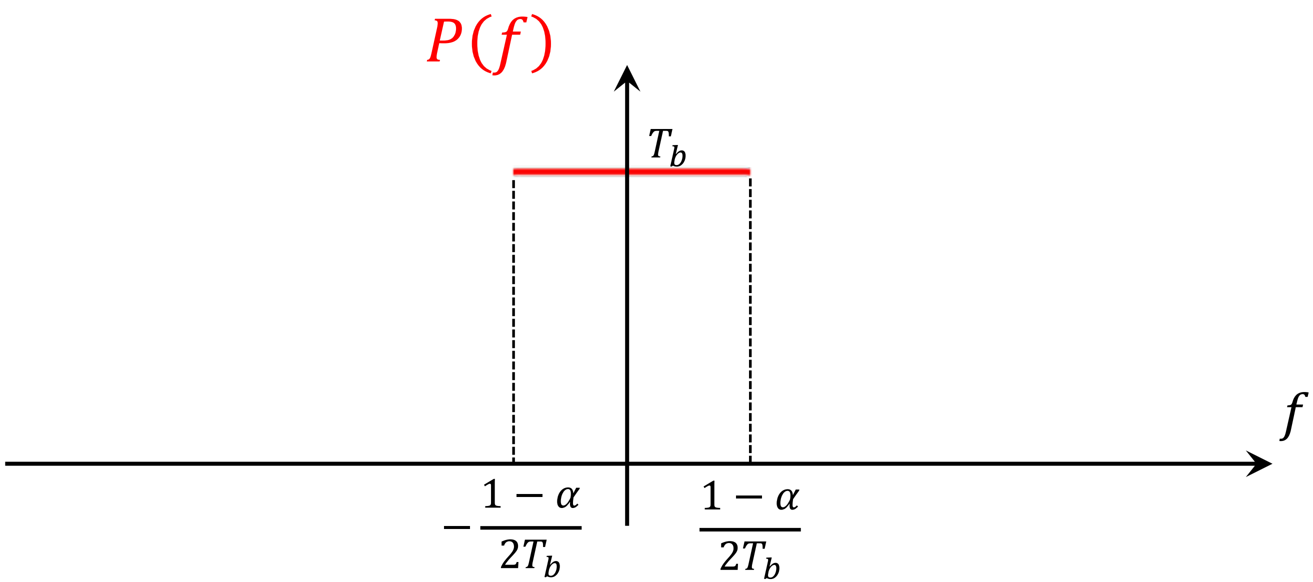
\+ No ISI if sampling is aligned

\+ Bandlimited in Frequency

Pulse Shaping: Raised Cosine

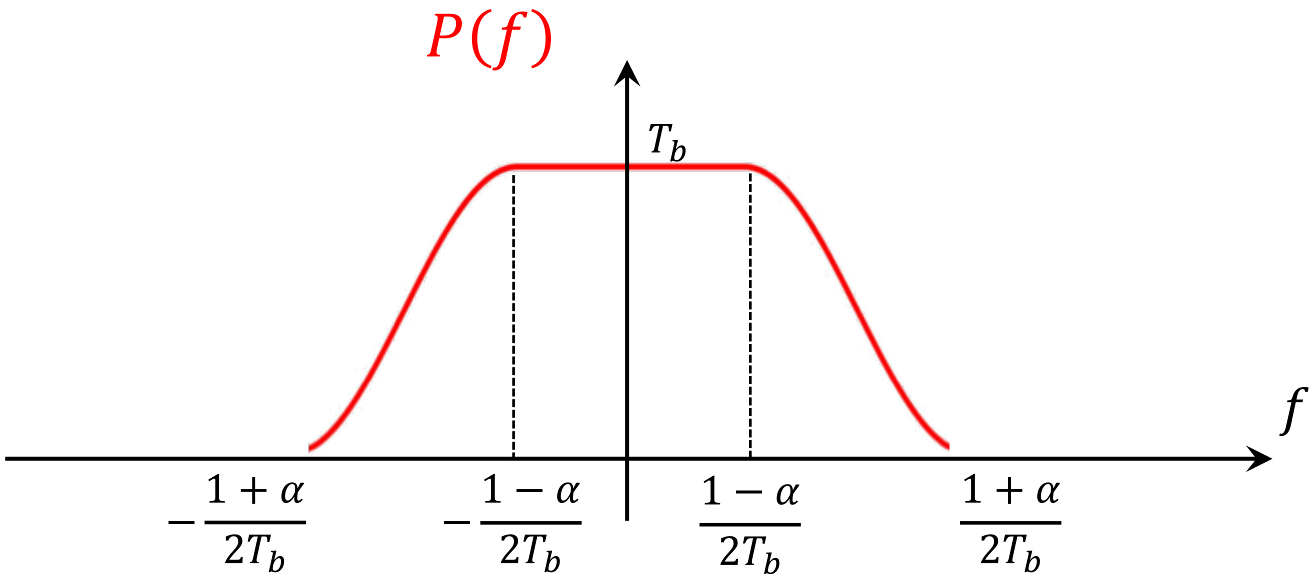
Pulse Shaping: Raised Cosine

$$P(f) = \begin{cases} T_b & |f| \leq \dfrac{1-\alpha}{2T_b} \\ & \\ & \end{cases}$$



Pulse Shaping: Raised Cosine

$$P(f) = \begin{cases} T_b & |f| \leq \dfrac{1-\alpha}{2T_b} \\ \dfrac{T_b}{2}\left[1 + \cos\left(\dfrac{\pi T_b}{\alpha}\left(|f| - \dfrac{1-\alpha}{2T_b}\right)\right)\right] & \dfrac{1-\alpha}{2T_b} \leq |f| \leq \dfrac{1+\alpha}{2T_b} \end{cases}$$



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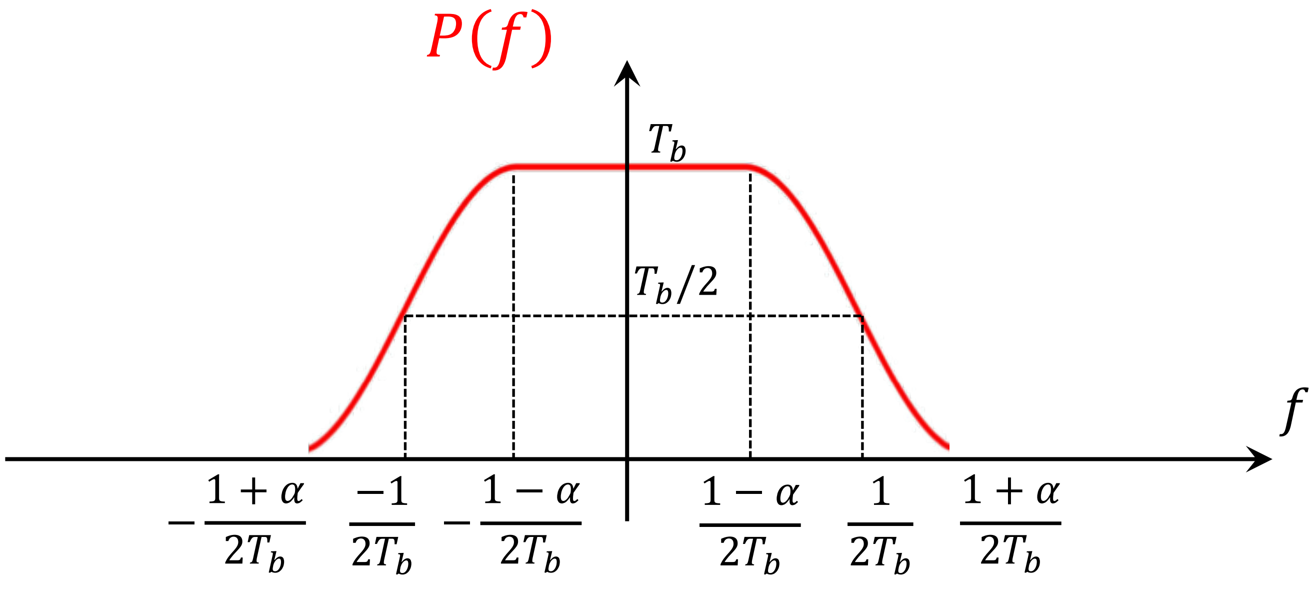

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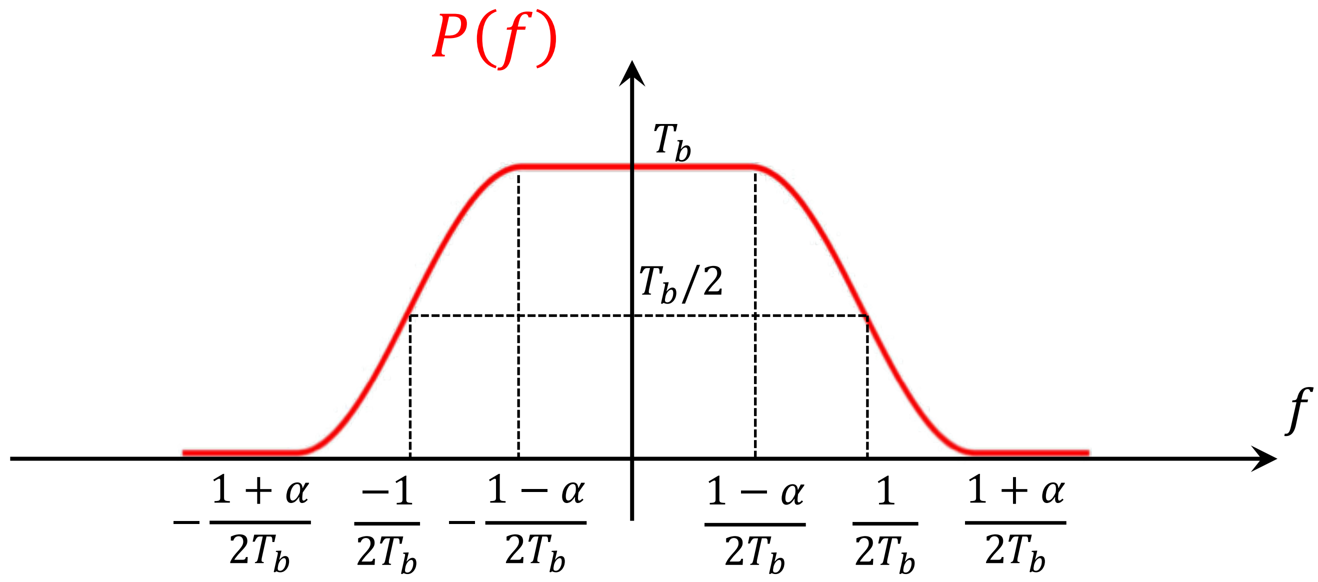

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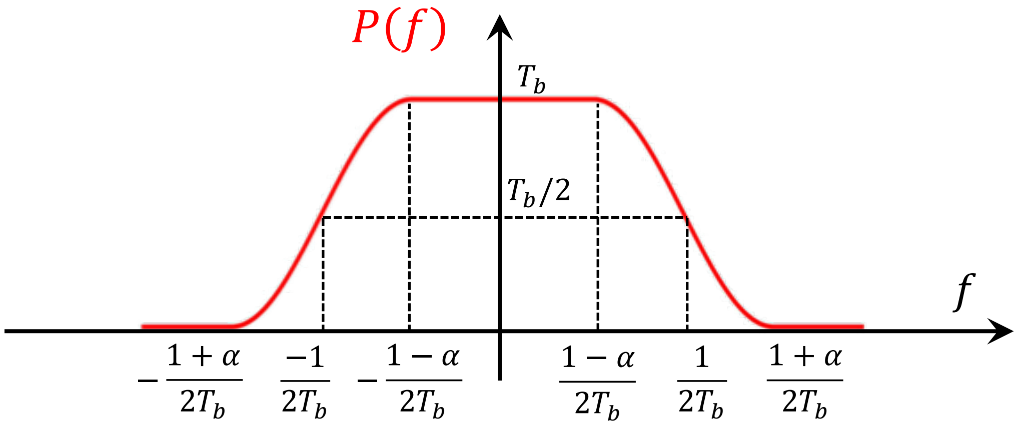

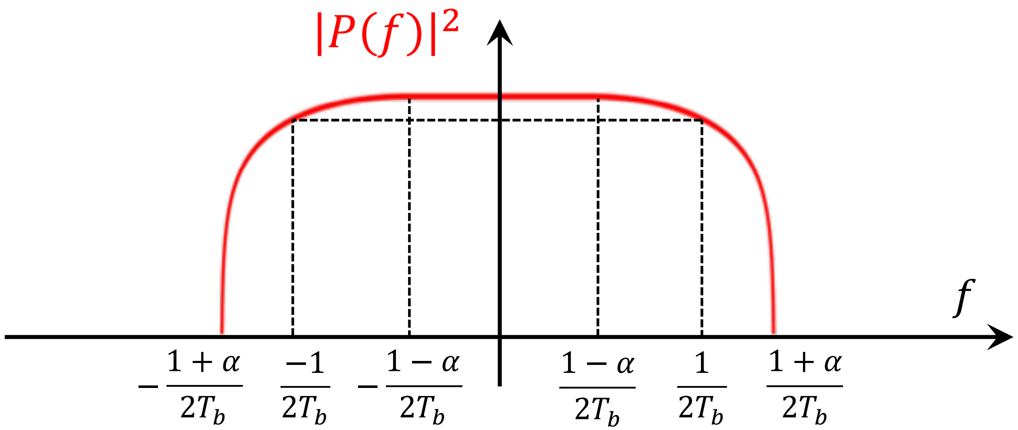

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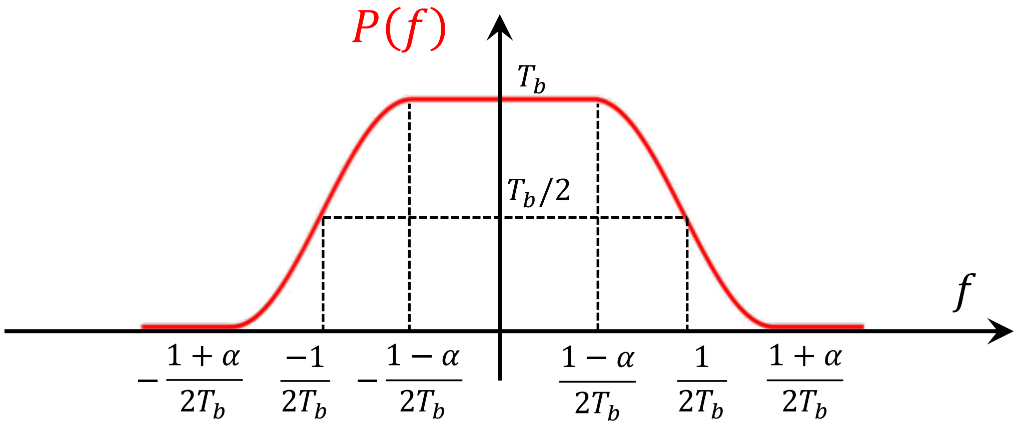

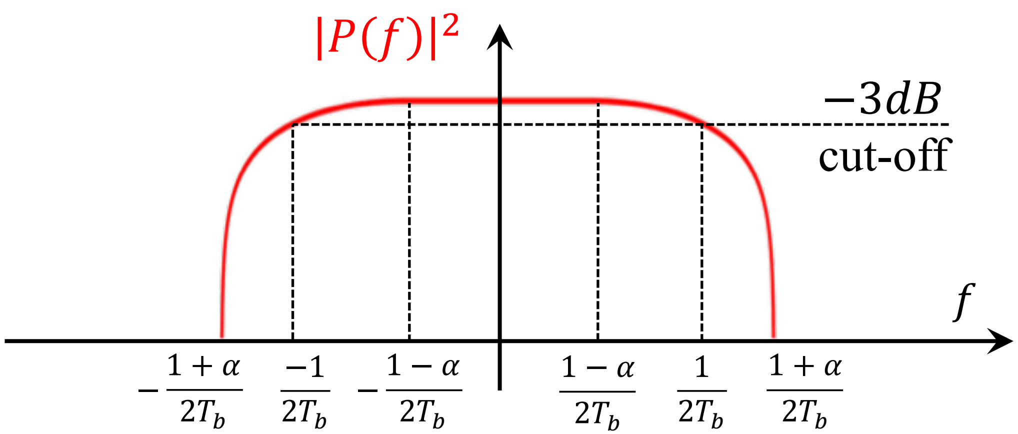

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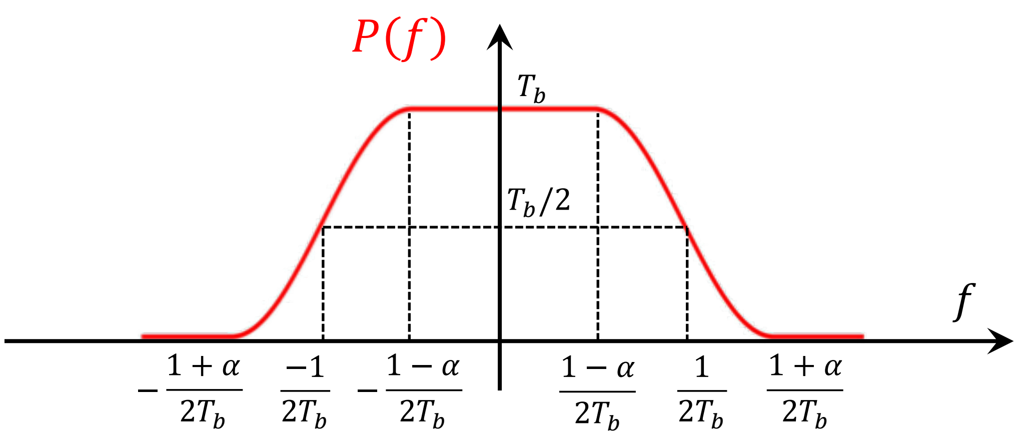

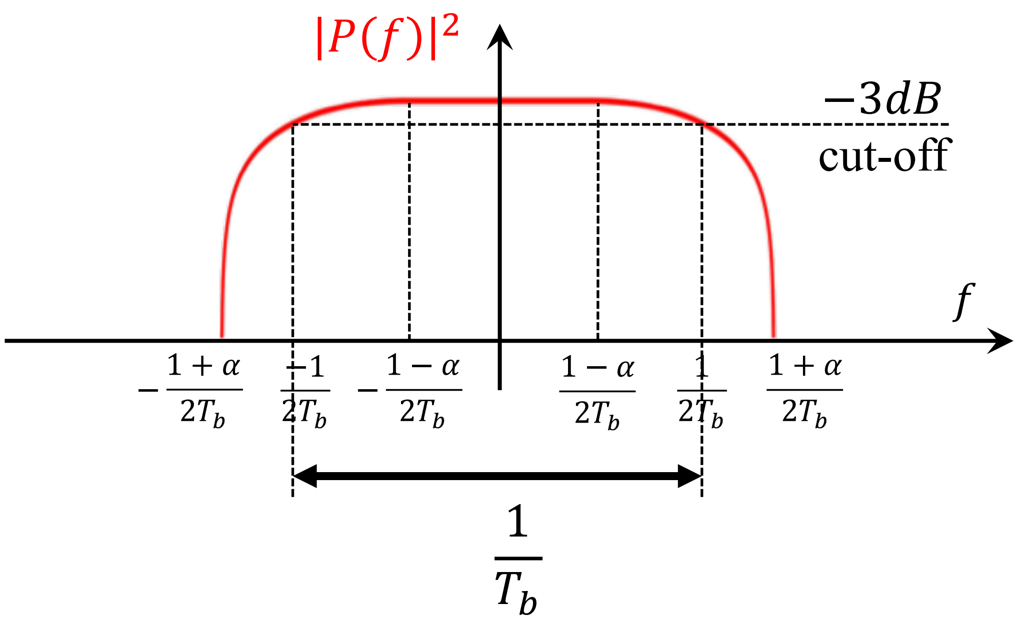

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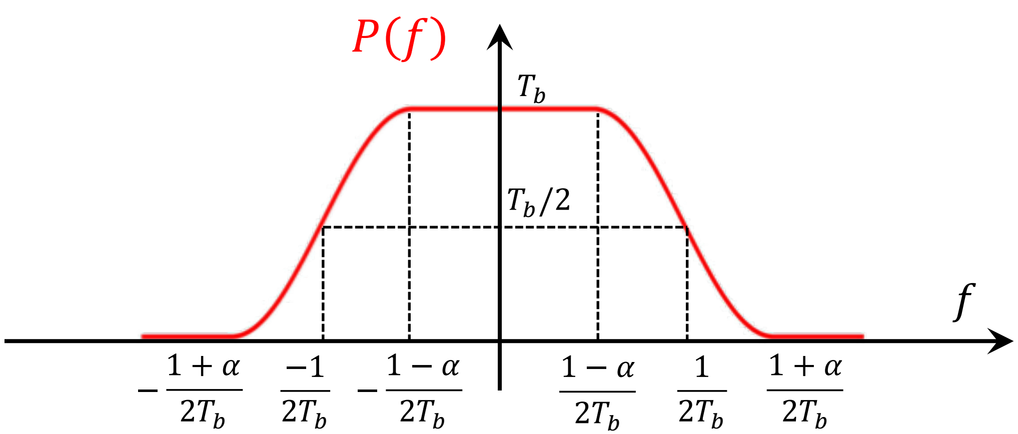

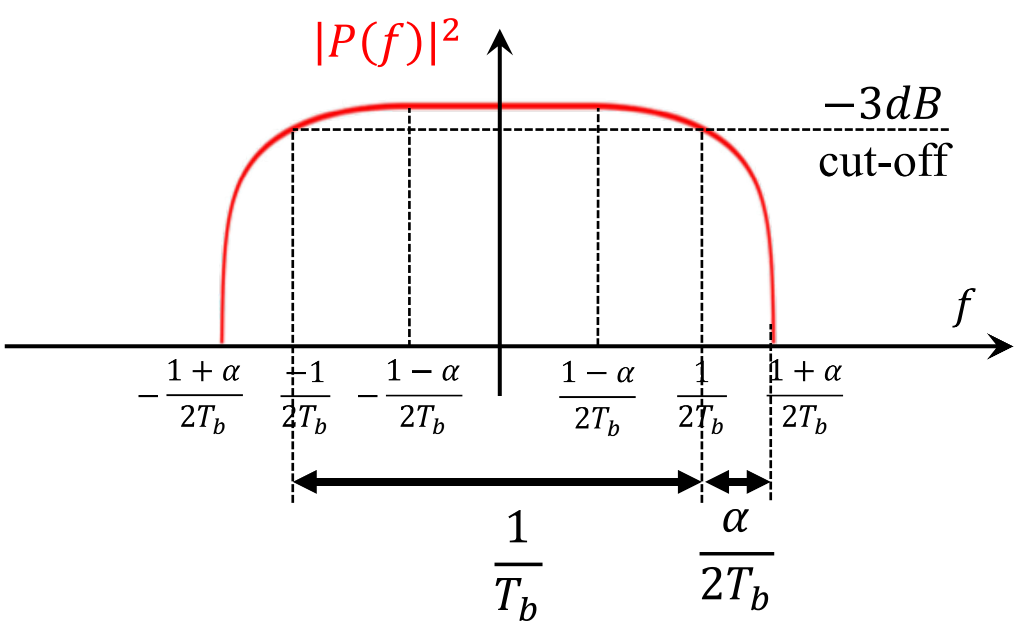

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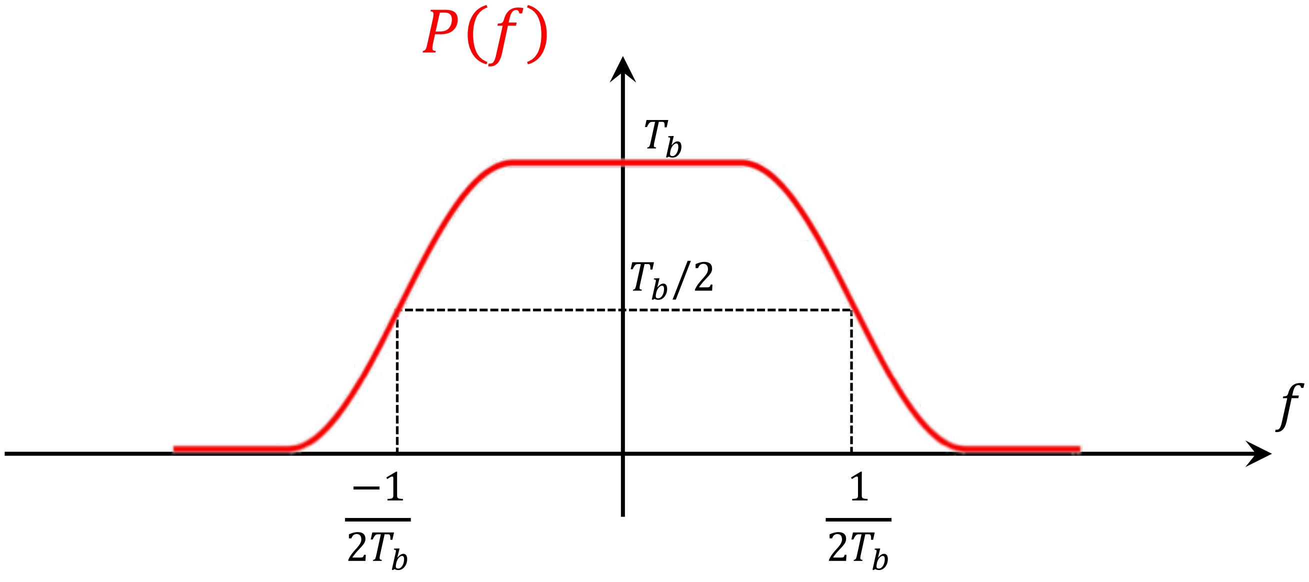

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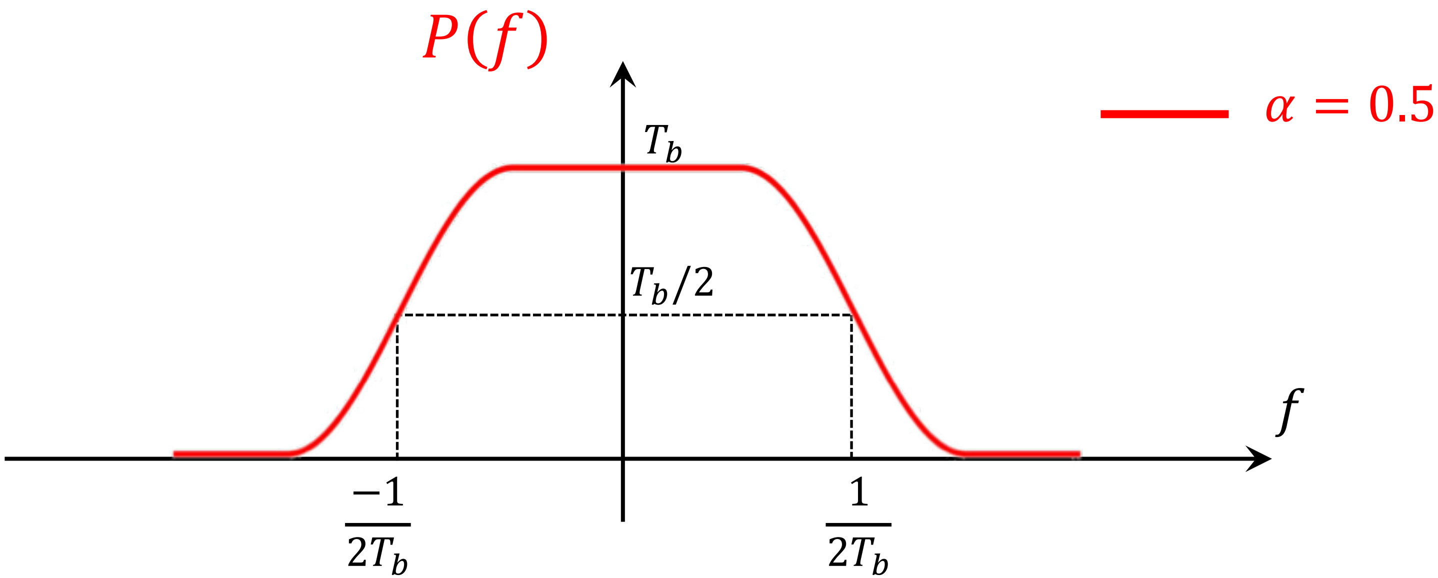
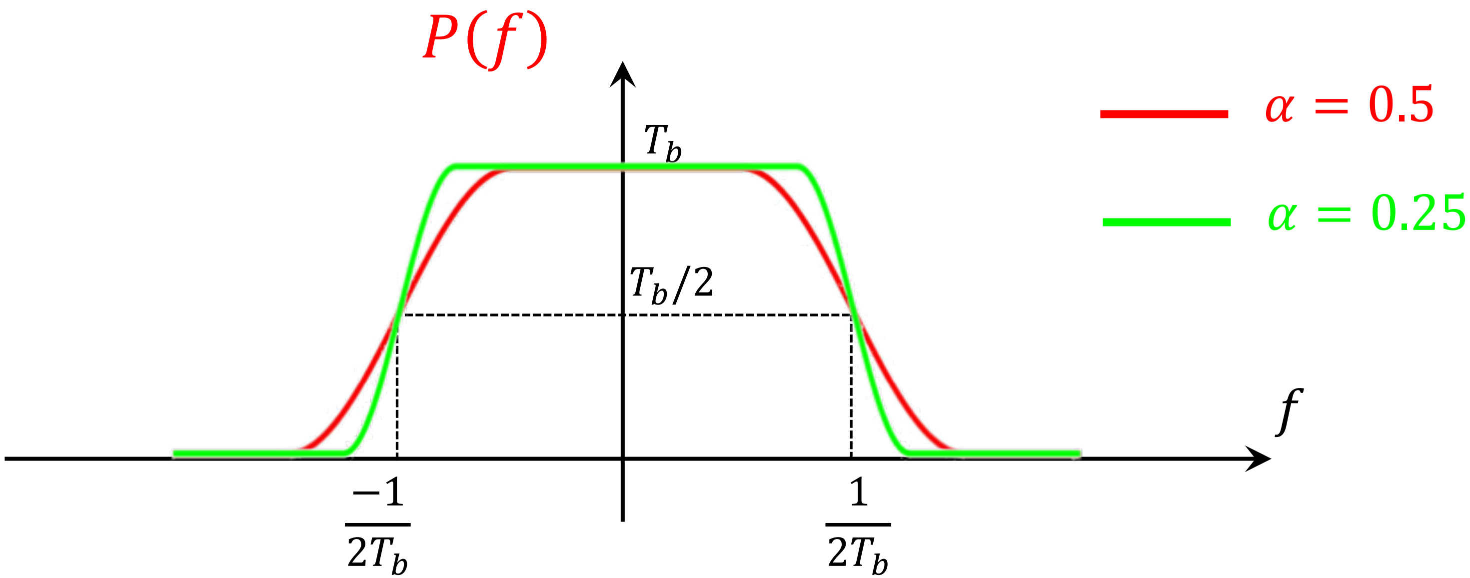


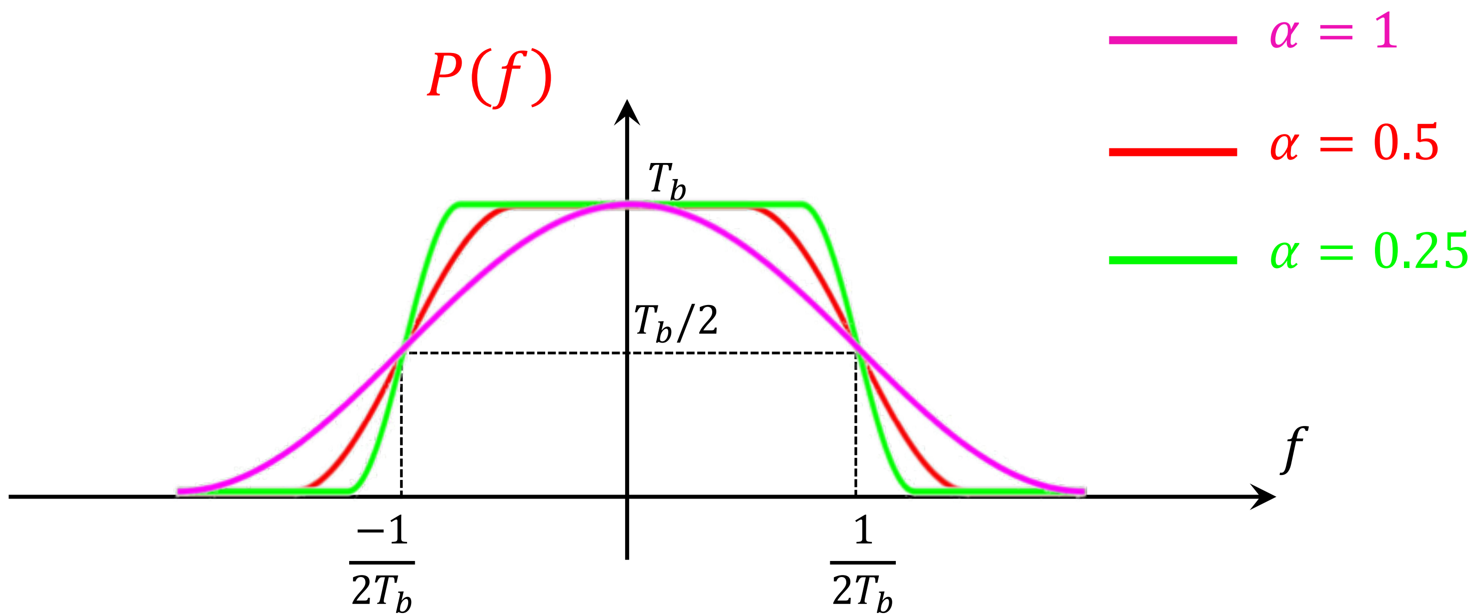
$$P(f) = \begin{cases} T_b & |f| \leq \frac{1}{4T_b} \\ \frac{T_b}{2}\left[1 + \cos\left(2\pi T_b|f| - \frac{\pi}{2}\right)\right] & \frac{1}{4T_b} \leq |f| \leq \frac{3}{4T_b} \\ 0 & \text{otherwise} \end{cases}$$

Pulse Shaping: Raised Cosine



$$P(f) = \begin{cases} T_b & |f| \leq \frac{3}{8T_b} \\ \frac{T_b}{2}\left[1 + \cos\left(4\pi T_b|f| - \frac{3\pi}{2}\right)\right] & \frac{3}{8T_b} \leq |f| \leq \frac{5}{8T_b} \\ 0 & \text{otherwise} \end{cases}$$

Pulse Shaping: Raised Cosine



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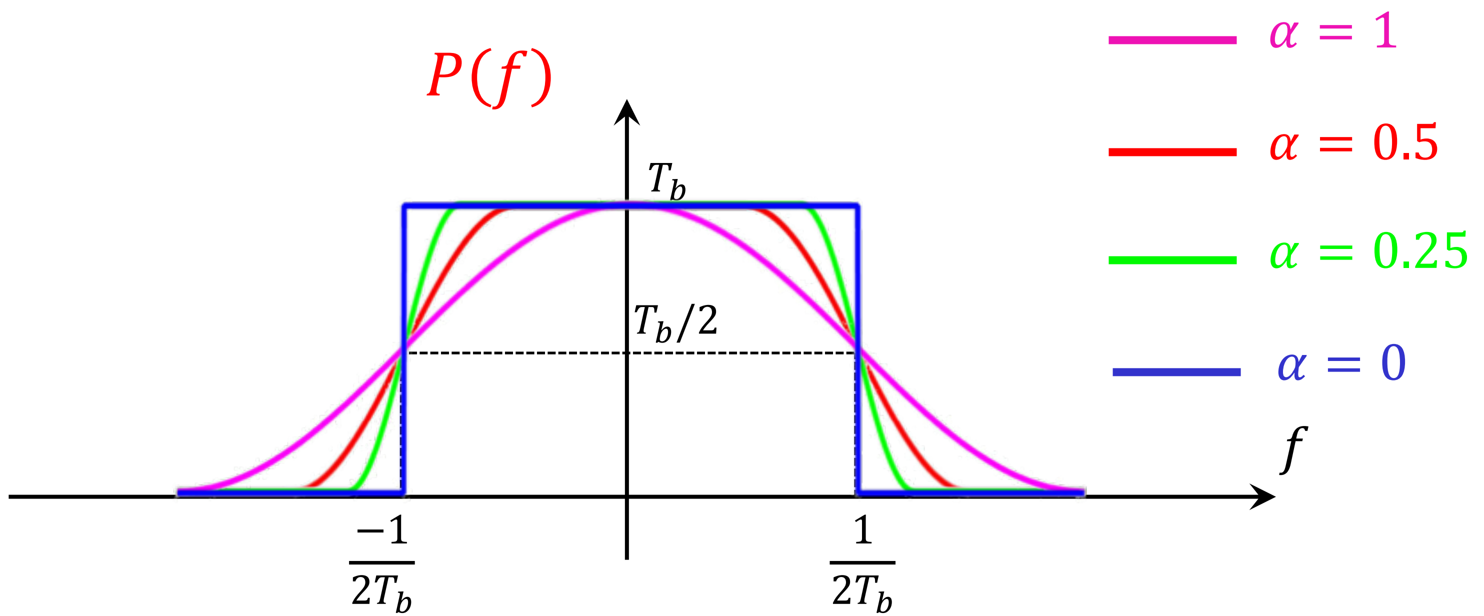

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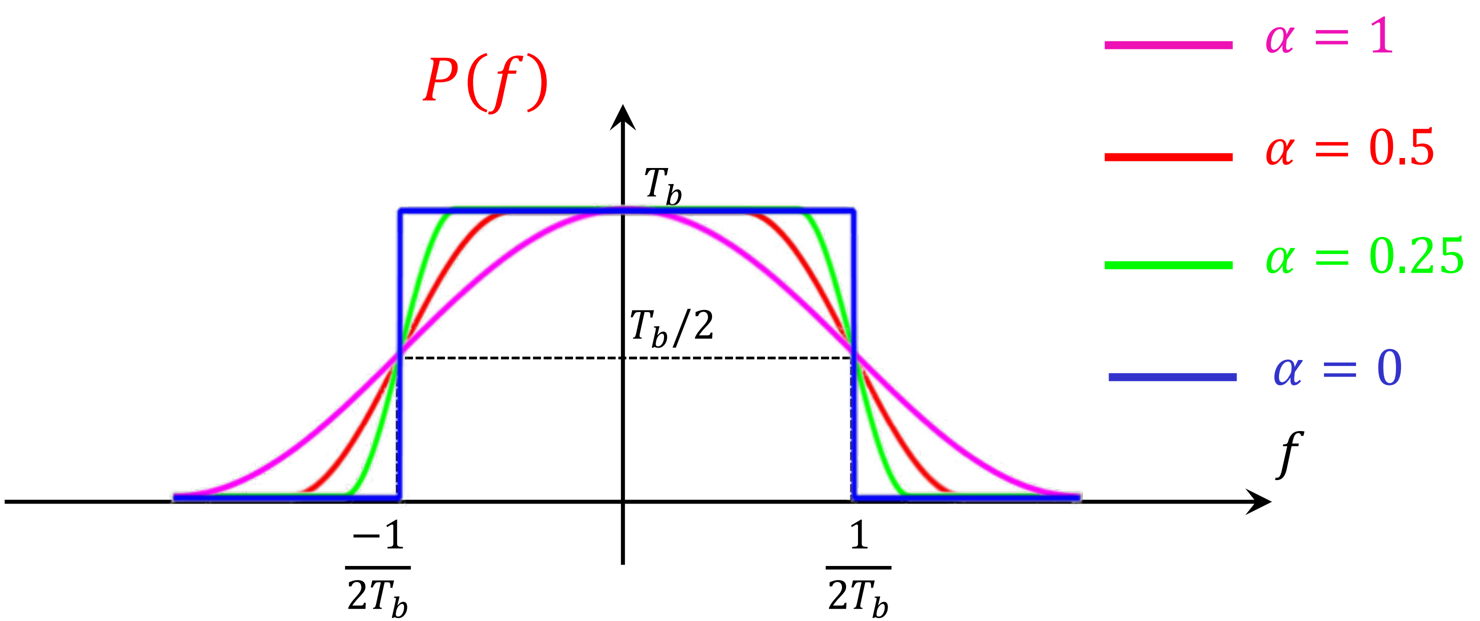
Pulse Shaping: Raised Cosine



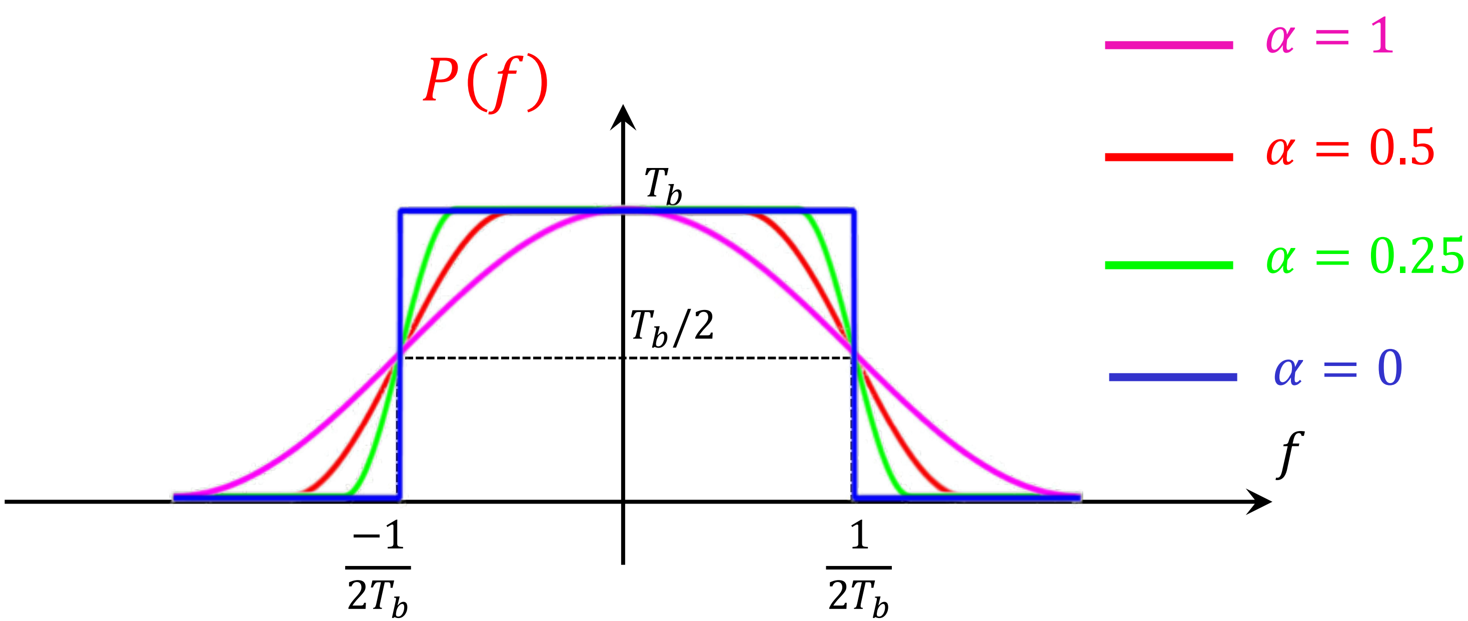
$$P(f) = \begin{cases} T_b & |f| \leq \frac{1-\alpha}{2T_b} \\ \frac{T_b}{2}\left[1 + \cos\left(\frac{\pi T_b}{\alpha}\left(|f| - \frac{1-\alpha}{2T_b}\right)\right)\right] & \frac{1-\alpha}{2T_b} \leq |f| \leq \frac{1+\alpha}{2T_b} \\ 0 & \text{otherwise} \end{cases}$$

α: Rolloff Factor

$\uparrow \alpha$: $\uparrow$ Bandwidth leakage

$\downarrow \alpha$: $\downarrow$ Bandwidth leakage

Pulse Shaping: Raised Cosine



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➡

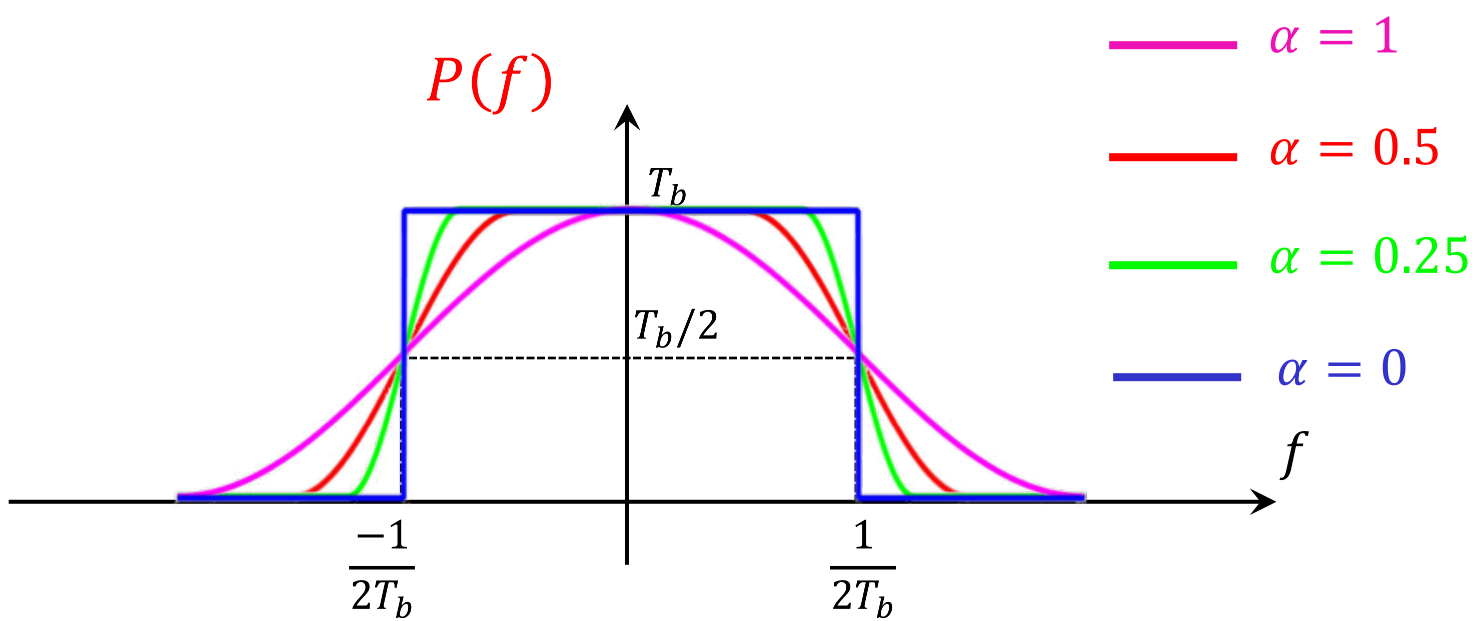
$$p(t) = \mathrm{sinc}(t/T_b)\,\frac{\cos(\pi\alpha t/T_b)}{1 - 4\,\alpha^2 t^2/T_b^2}$$

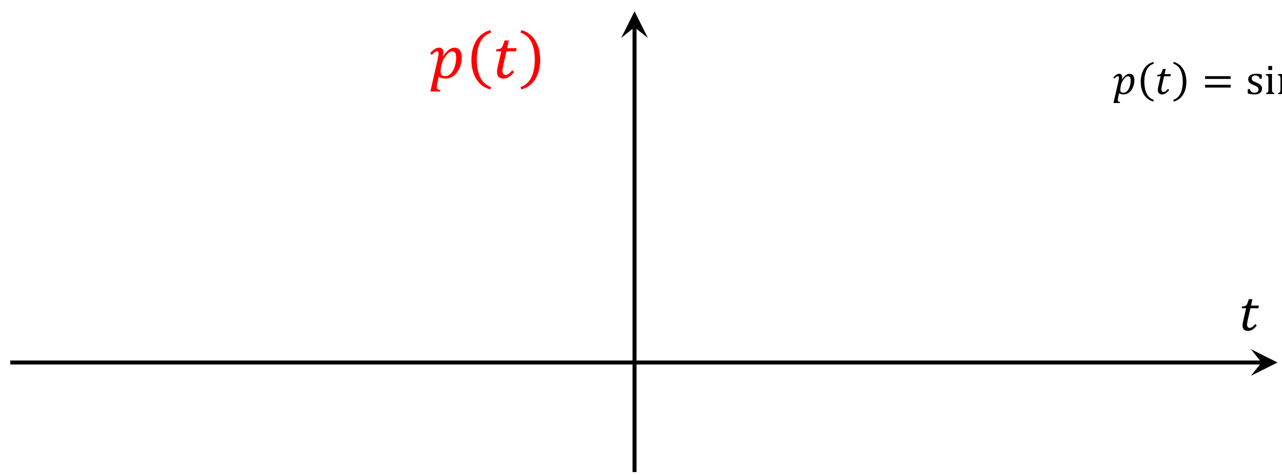
α: Rolloff Factor

↑ α : ↑ Bandwidth leakage

↓ α : ↓ Bandwidth leakage

Pulse Shaping: Raised Cosine





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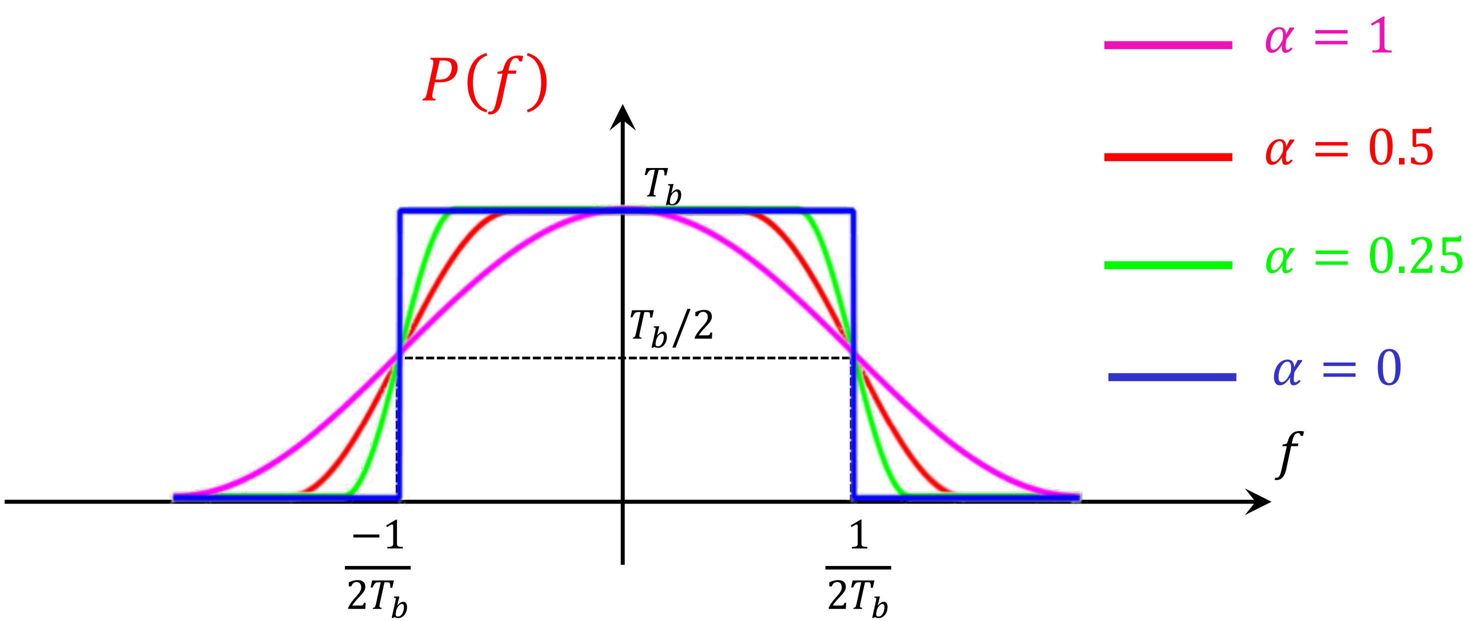
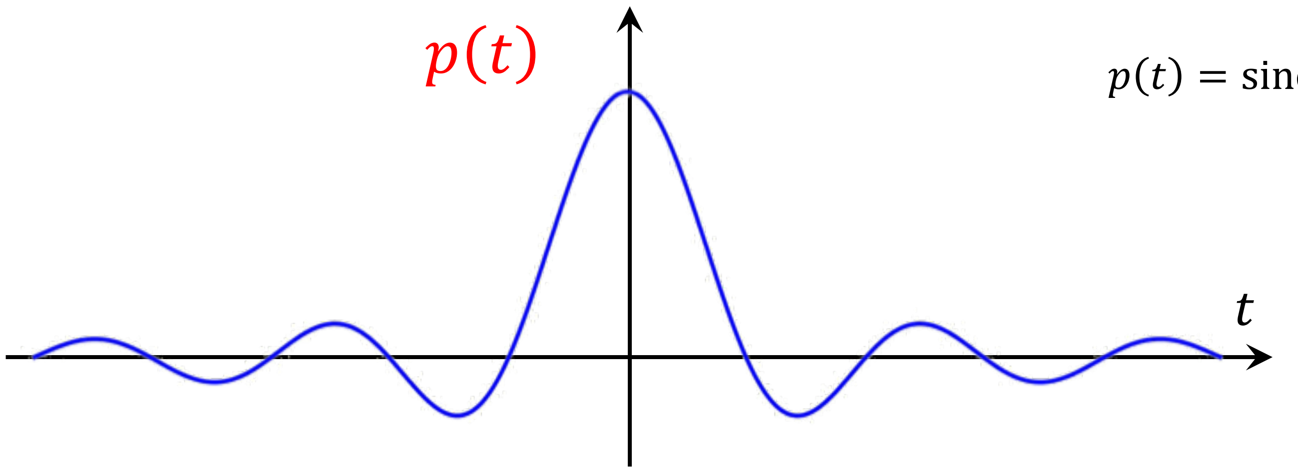
↓ α : ↓ Bandwidth leakage

Pulse Shaping: Raised Cosine





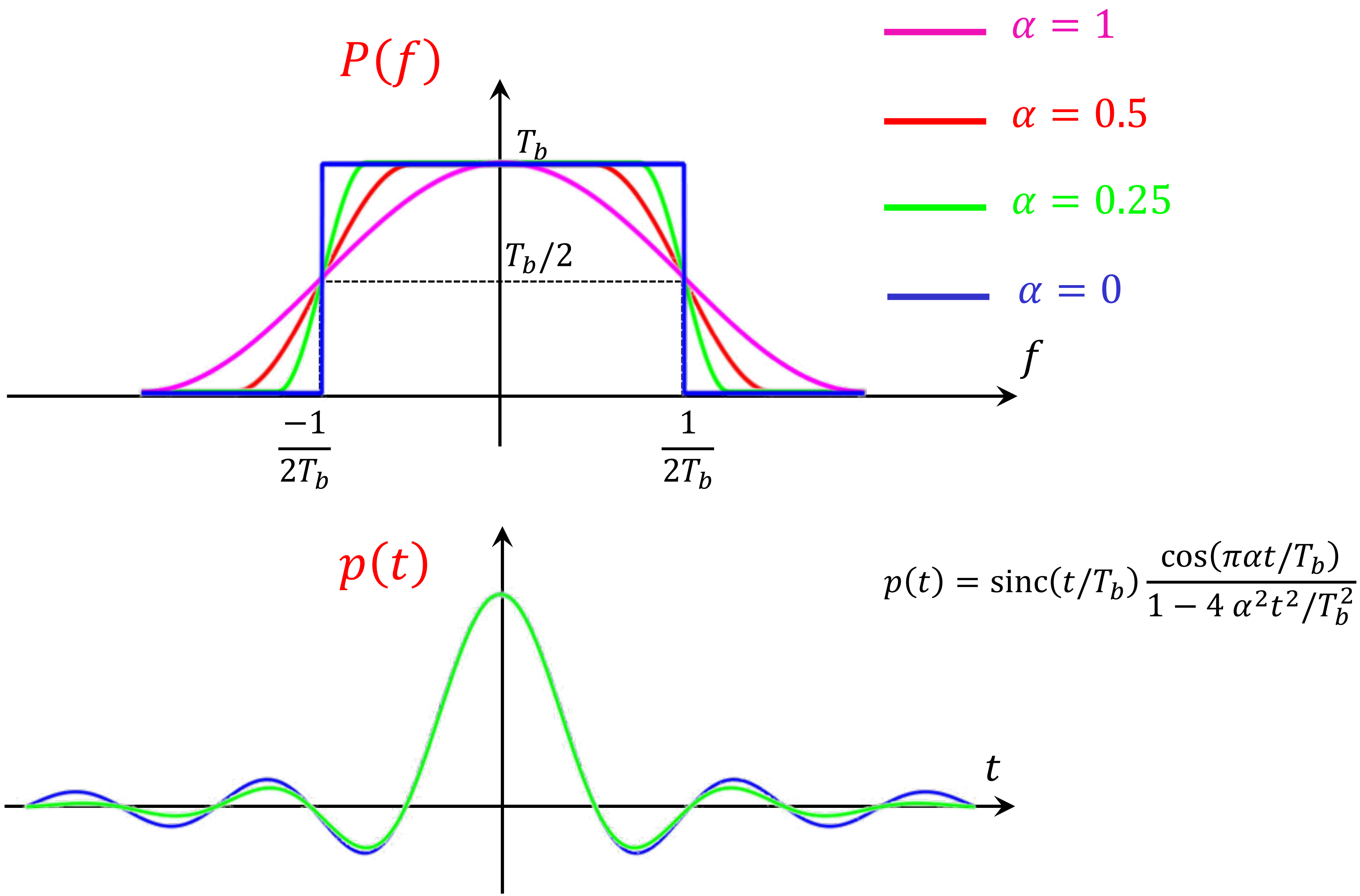
$$p(t) = \mathrm{sinc}(t/T_b)\,\frac{\cos(\pi\alpha t/T_b)}{1 - 4\,\alpha^2 t^2/T_b^2}$$

α: Rolloff Factor

$\uparrow \alpha$: $\uparrow$ Bandwidth leakage

$\downarrow \alpha$: $\downarrow$ Bandwidth leakage

Pulse Shaping: Raised Cosine



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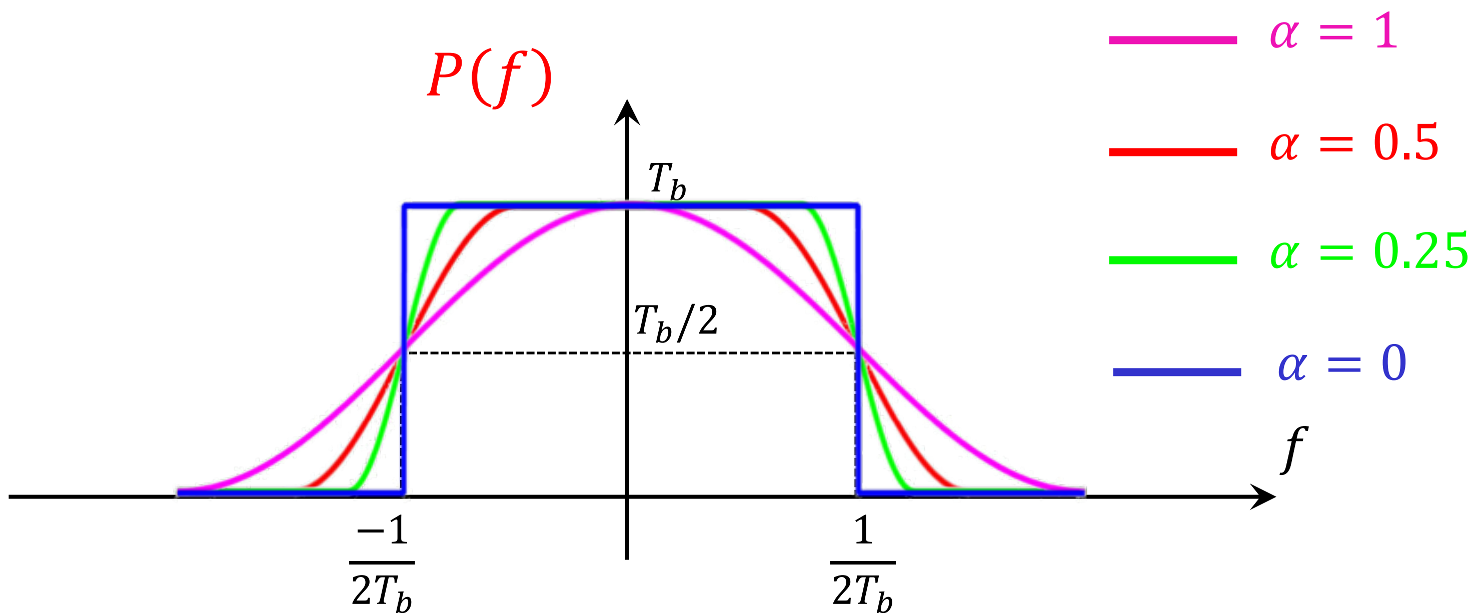

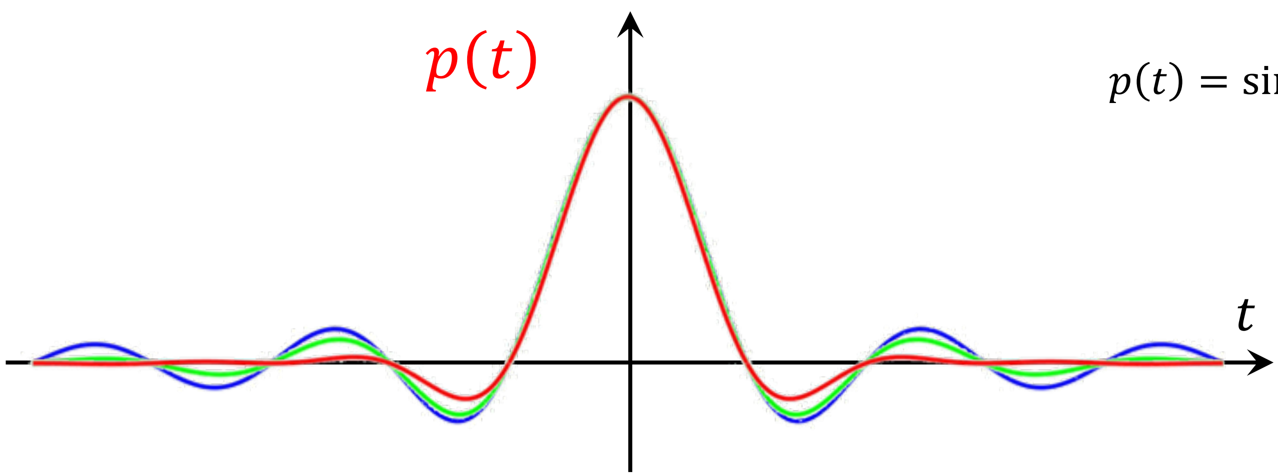

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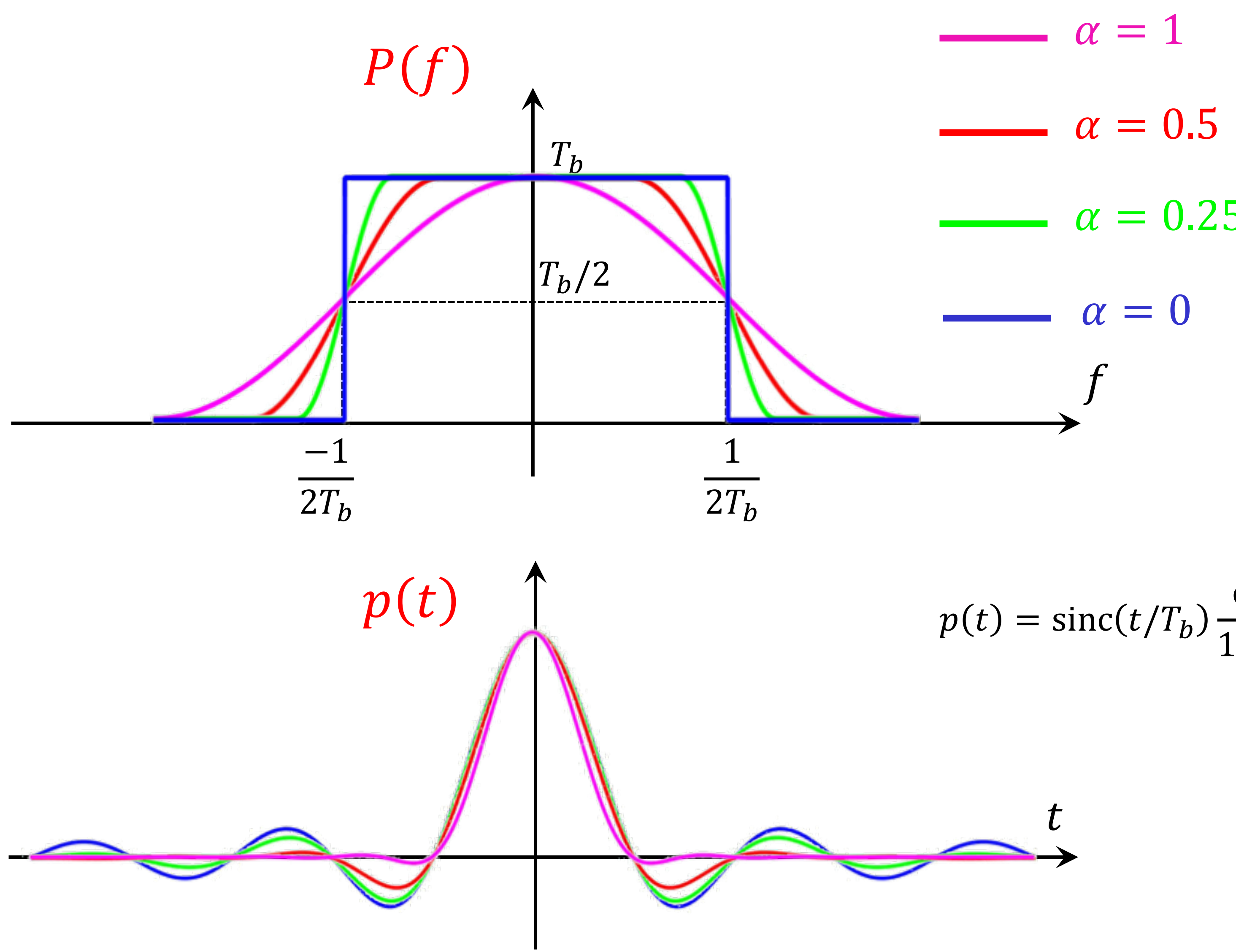
Pulse Shaping: Raised Cosine



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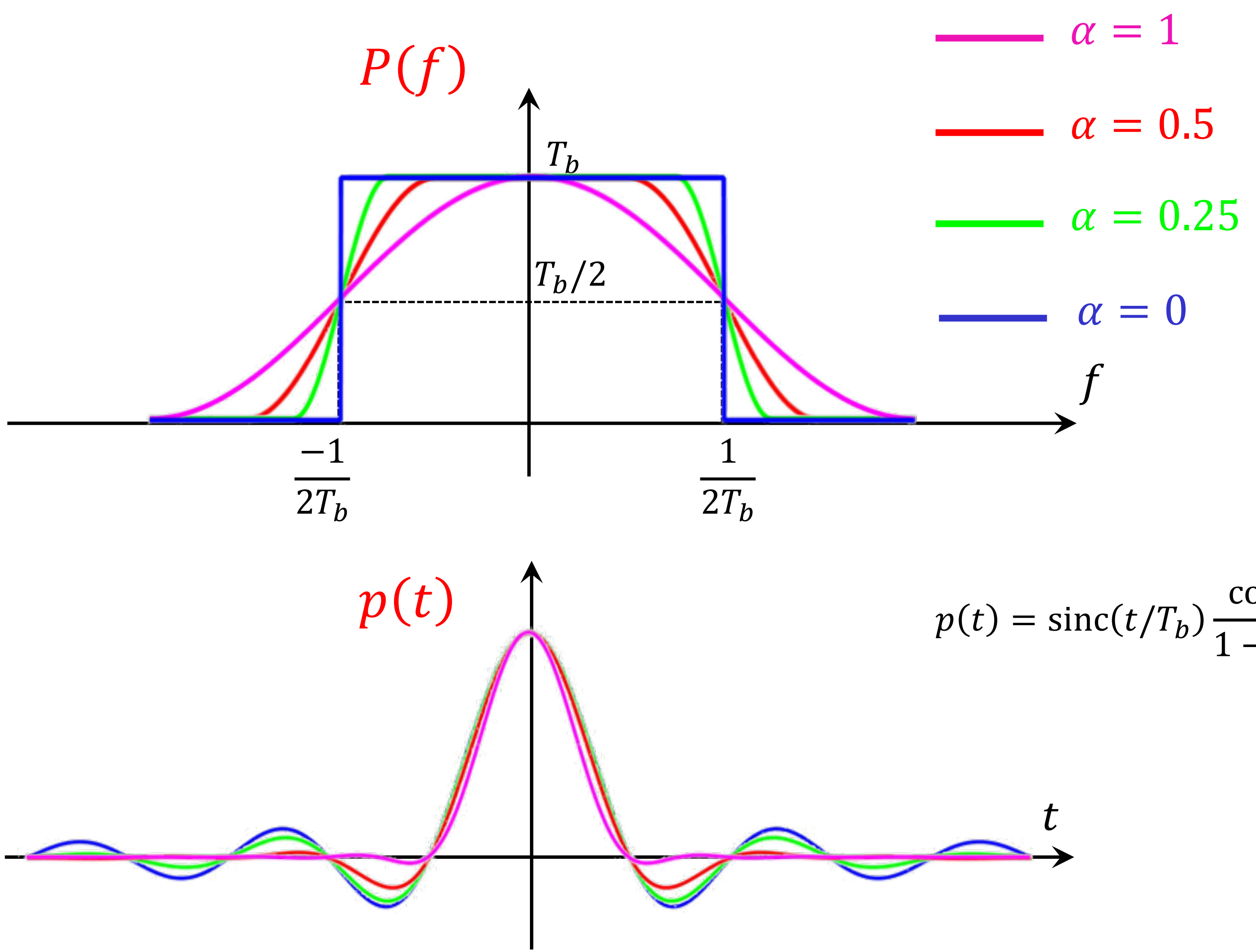
Pulse Shaping: Raised Cosine



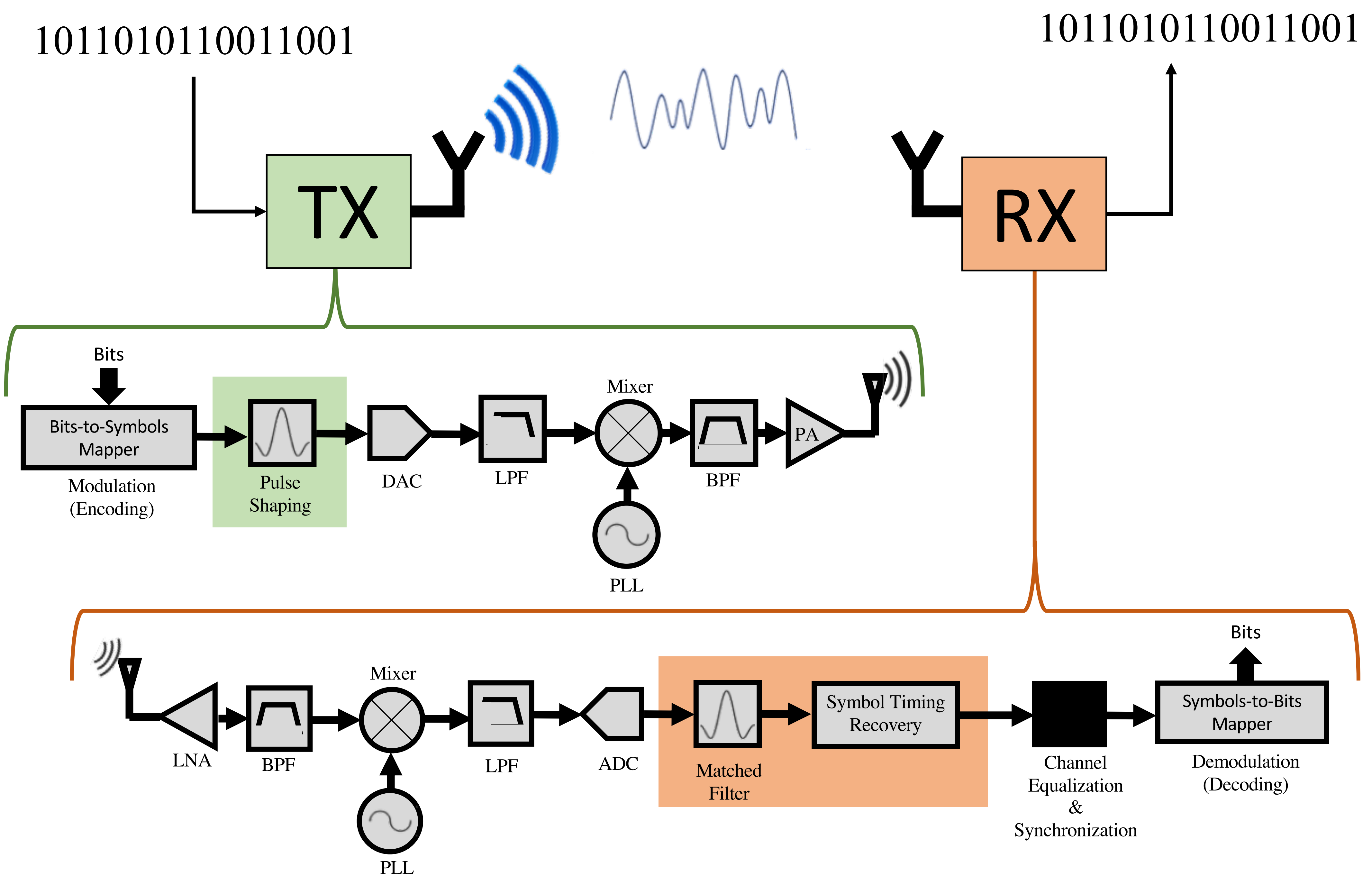
$$p(t) = \text{sinc}(t/T_b)\frac{\cos(\pi\alpha t/T_b)}{1 - 4\,\alpha^2 t^2/T_b^2}$$

α: Rolloff Factor

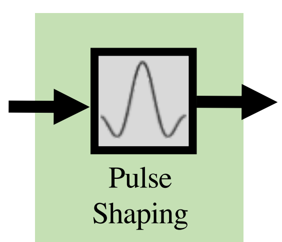
↑ α : ↑ Bandwidth leakage, ↓ Time Support, ↓ Sidelobes → ↓ ISI if sampling not aligned

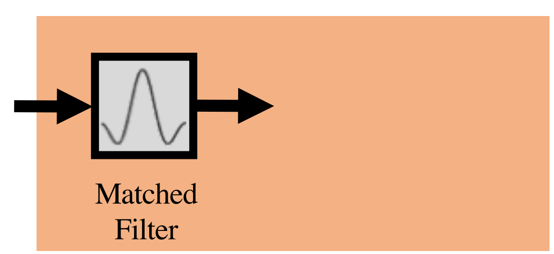
↓ α : ↓ Bandwidth leakage, ↑ Time Support, ↑ Sidelobes → ↑ ISI if sampling not aligned

Digital Communication System

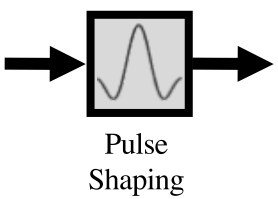


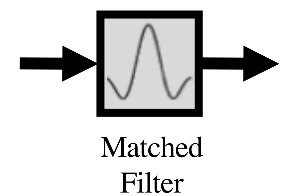
Digital Communication System





Pulse Shaping and Matched Filtering





Pulse Shaping and Matched Filtering

Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = \sum_{n=-\infty}^{+\infty} s[n]p(t - nT_b)$$

Let us focus on a single symbol i.e., single n

Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = s(t) * p(t)$$

$$s(t) = s[n]\delta(t - nT_b)$$

Pulse Shaping and Matched Filtering

$s[n] \Rightarrow x(t) = s(t) * p_T(t)$

$$s(t) = s[n]\delta(t - nT_b)$$

Pulse Shaping and Matched Filtering

$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t)$

Consider Simple AWGN Channel.
$v(t)$ is Gaussian noise.

Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$$

$$\tilde{y}(t) = s[n]\delta(t - nT_b) * p_T(t) * p_R(t) + v(t) * p_R(t)$$

Pulse Shaping and Matched Filtering

$s[n]$ ➡ $x(t) = s(t) * p_T(t)$ ➡ $y(t) = x(t) + v(t)$ ➡ $\tilde{y}(t) = y(t) * p_R(t)$

$$\tilde{y}(t) = s[n]\delta(t - nT_b) * p_T(t) * p_R(t) + v(t) * p_R(t)$$

Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$$

$$\tilde{y}(t) = \underbrace{s[n]\delta(t - nT_b) * p_T(t) * p_R(t)}_{\tilde{y}_s(t)} + \underbrace{v(t) * p_R(t)}_{\tilde{y}_v(t)}$$

$$\tilde{Y}_s(f) = s[n]e^{-j2\pi nT_b f} \cdot P_T(f) \cdot P_R(f)$$

$$\tilde{y}_s(t) = \mathcal{F}^{-1}\{\tilde{Y}_s(f)\} = \int \tilde{Y}_s(f)e^{j2\pi tf}\, df$$

$$\tilde{y}_s(nT_b) = \int \tilde{Y}_s(f)e^{j2\pi nT_b f}\, df = \int s[n]e^{-j2\pi nT_b f} P_T(f)P_R(f)\, e^{j2\pi nT_b f}\, df$$

Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$$

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Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$$

$$\tilde{y}(t) = \underbrace{s[n]\delta(t - nT_b) * p_T(t) * p_R(t)}_{\tilde{y}_s(t)} + \underbrace{v(t) * p_R(t)}_{\tilde{y}_v(t)}$$

$$\tilde{y}_s(nT_b) = s[n] \int P_T(f) P_R(f)\ df$$

$$E[|\tilde{y}_v(nT_b)|^2] = E[|v(t) * p_R(t)|^2]$$

$$= \int |V(f) P_R(f)|^2 df$$

$$= \frac{N_0}{2} \int |P_R(f)|^2 df$$

$$SNR = \frac{|\tilde{y}_s(nT_b)|^2}{E[|\tilde{y}_v(nT_b)|^2]}$$

Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$$

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$$\tilde{y}_s(nT_b) = s[n] \int P_T(f) P_R(f)\ df \qquad E[|\tilde{y}_v(nT_b)|^2] = \frac{N_0}{2} \int |P_R(f)|^2 df$$

$$SNR = \frac{|\tilde{y}_s(nT_b)|^2}{E[|\tilde{y}_v(nT_b)|^2]} = \frac{|s[n]|^2 \left| \int P_T(f) P_R(f) \right|^2}{\frac{N_0}{2} \int |P_R(f)|^2}$$

GOAL: Find $P_R(f)$ that maximizes the SNR

Pulse Shaping and Matched Filtering

$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$

GOAL: Find $P_R(f)$ that maximizes the SNR

$$SNR = \frac{|s[n]|^2 \left| \int P_T(f) P_R(f) \right|^2}{\frac{N_0}{2} \int |P_R(f)|^2}$$

Cauchy-Schwarz Inequality :

$$\left| \int P_T(f) P_R(f) \right|^2 \leq \int |P_T(f)|^2 \cdot \int |P_R(f)|^2$$

$$= \int |P_T(f)|^2 \cdot \int |P_R(f)|^2 \quad if \quad P_R(f) = C \cdot P_T^*(f)$$

Pulse Shaping and Matched Filtering

$$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$$

GOAL: Find $P_R(f)$ that maximizes the SNR

$$SNR = \frac{|s[n]|^2 \left| \int P_T(f) P_R(f) \right|^2}{\frac{N_0}{2} \int |P_R(f)|^2}$$

SNR is maximized when: $P_R(f) = C \cdot P_T^*(f) \Rightarrow p_R(t) = C \cdot p_T^*(-t)$

$p_R(t)$ matches $p_T(t)$… hence, the name matched filter !

$$SNR = \frac{|s[n]|^2 \int |P_T(f)|^2 \int |P_R(f)|^2}{\frac{N_0}{2} \int |P_R(f)|^2} = \frac{2|s[n]|^2}{N_0}$$

Pulse Shaping and Matched Filtering

$s[n]$ ➡ $x(t) = s(t) * p_T(t)$ ➡ $y(t) = x(t) + v(t)$ ➡ $\tilde{y}(t) = y(t) * p_R(t)$

$$\tilde{y}(t) = s[n]\delta(t - nT_b) * p_T(t) * p_R(t) + v(t) * p_R(t)$$

Pulse Shaping and Matched Filtering

$$s[n] \;\Rightarrow\; x(t) = s(t) * p_T(t) \;\Rightarrow\; y(t) = x(t) + v(t) \;\Rightarrow\; \tilde{y}(t) = y(t) * p_R(t)$$

$$\tilde{y}(t) = s[n]\delta(t - nT_b) * \underbrace{p_T(t) * p_T^*(-t)}_{p(t)} + v(t) * p_T^*(-t)$$

$p(t)$ should satisfy Nyquist Criterion for ISI

- Let $p(t)$ be a raised cosine
- What is $p_T(t)$?

$$P(f) = \mathcal{F}\{p(t)\} = \mathcal{F}\{p_T(t) * p_T^*(-t)\} = P_T(f) \cdot P_T^*(f) = |P_T(f)|^2 = P_{rc}(f)$$

$$|P_T(f)| = \sqrt{P_{rc}(f)} = P_{srrc}(f)$$ ➡ Use square-root of raised cosine filter

Pulse Shaping and Matched Filtering

$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$

$$\tilde{y}(t) = s[n]\delta(t - nT_b) * \underbrace{p_T(t) * p_T^*(-t)}_{p(t)} + v(t) * p_T^*(-t)$$

$p(t)$ should satisfy Nyquist Criterion for ISI

- Let $p(t)$ be a raised cosine
- What is $p_T(t)$?

Square-root of raised cosine filter: $P_{srrc}(f) = \sqrt{P_{rc}(f)}$

$$p_{srrc}(t) = -\frac{1}{\sqrt{T_b}} \frac{\sin\left((1-\alpha)\frac{\pi t}{T_b}\right) + \frac{4\alpha t}{T_b}\cos\left((1+\alpha)\frac{\pi t}{T_b}\right)}{\frac{\pi t}{T_b}\left(1 - \left(\frac{4\alpha t}{T_b}\right)^2\right)}$$

Symbol Timing Recovery

$$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$$

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$$\tilde{y}(t) = s[n]p(t - nT_b)$$

Symbol Timing Recovery

$s[n]$ ➡ $x(t) = s(t) * p_T(t)$ ➡ $y(t) = x(t) + v(t)$ ➡ $\tilde{y}(t) = y(t) * p_R(t)$

$$\tilde{y}(t) = \sum_n s[n]p(t - nT_b)$$

Sample signal at $t = mT_b$

$$\tilde{y}(mT_b) = \sum_n s[n]p(mT_b - nT_b) = s[m] \text{ (NO ISI)}$$

$p(t)$ satisfies Nyquist → $p(mT_b - nT_b) = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

Symbol Timing Recovery

$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$

$$\tilde{y}(t) = \sum_n s[n]p(t - nT_b)$$

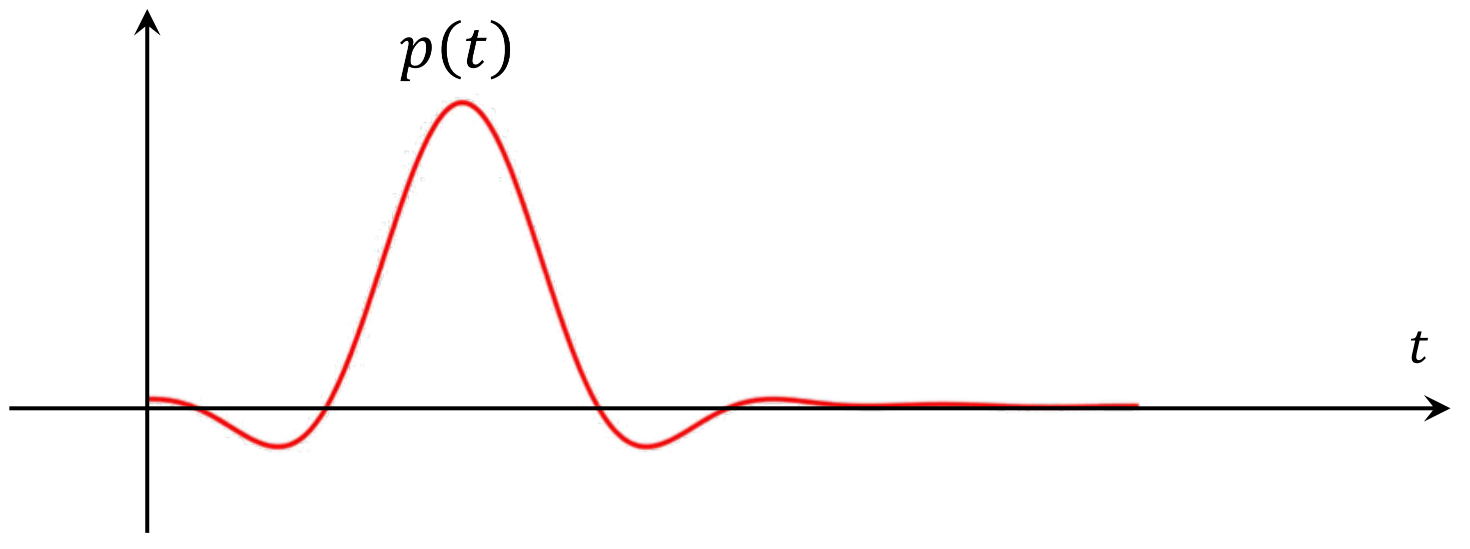
In practice, we have sampling offset:

Sample signal at $t = mT_b + \tau$

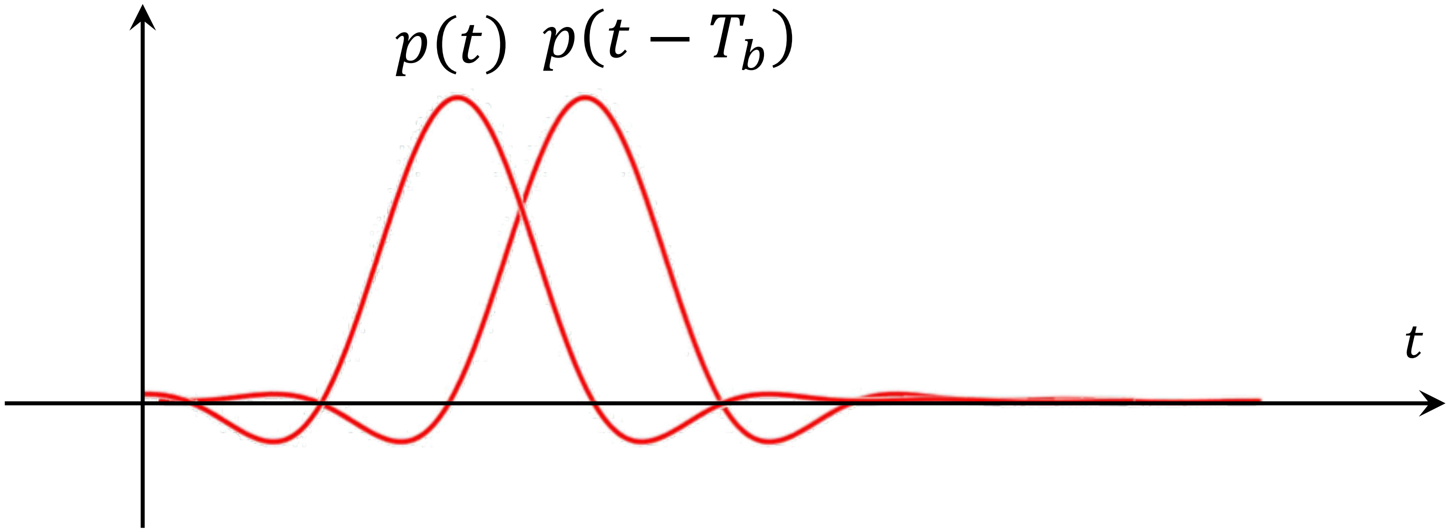
$$\tilde{y}(mT_b + \tau) = \sum_n s[n]p(mT_b - nT_b + \tau) \quad \neq s[m] \text{ (ISI)}$$

$$= s[m]\underbrace{p(\tau)}_{<1} + \underbrace{\sum_{n \neq m} s[n]p(mT_b - nT_b + \tau)}_{\text{ISI}}$$

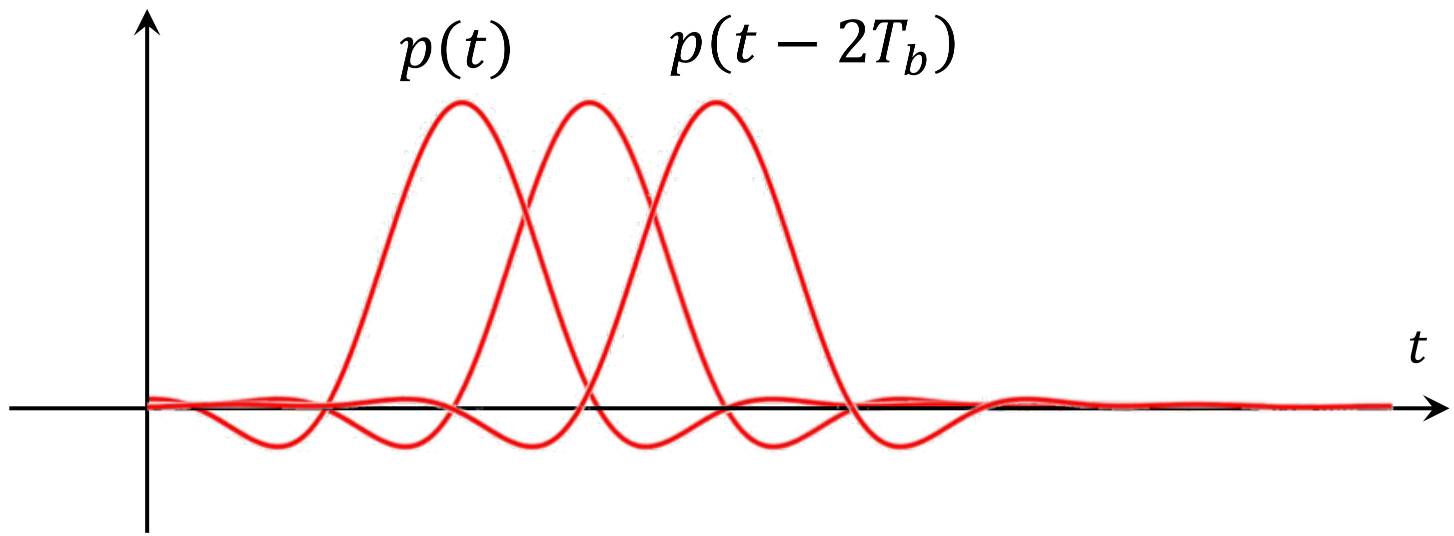
Symbol Timing Recovery



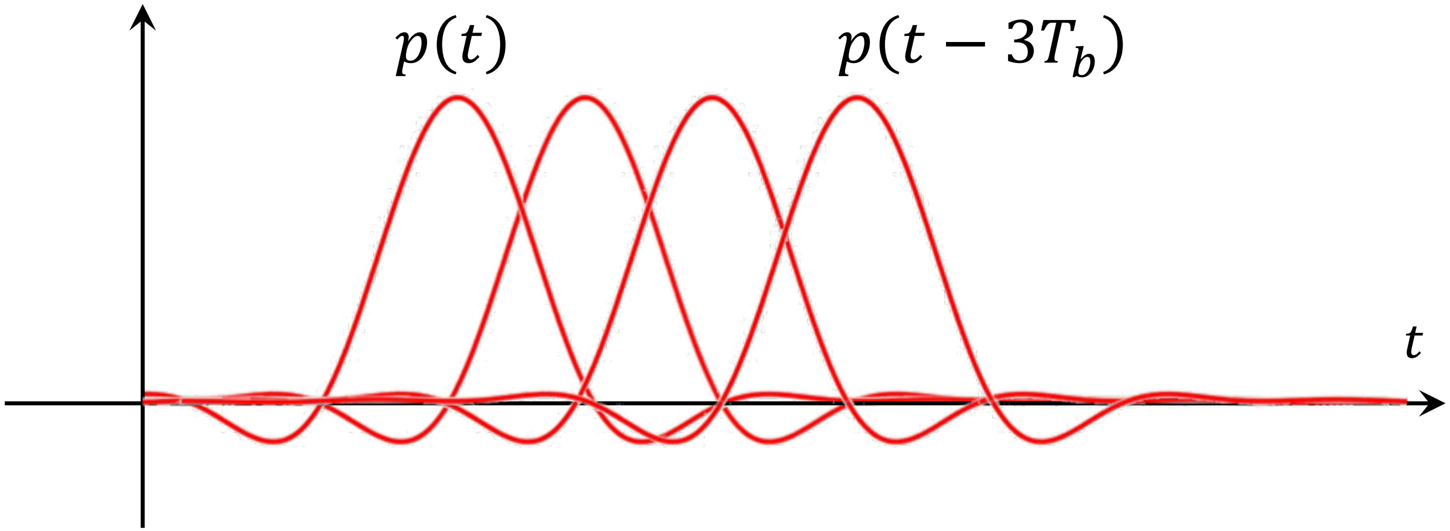
Symbol Timing Recovery



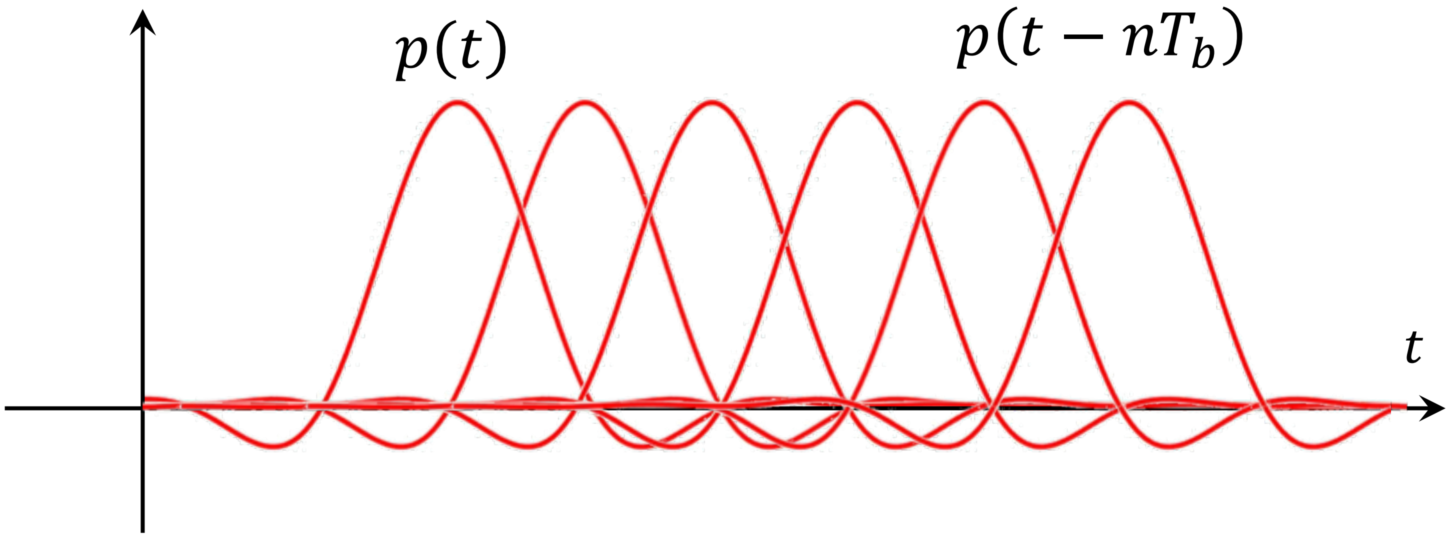
Symbol Timing Recovery



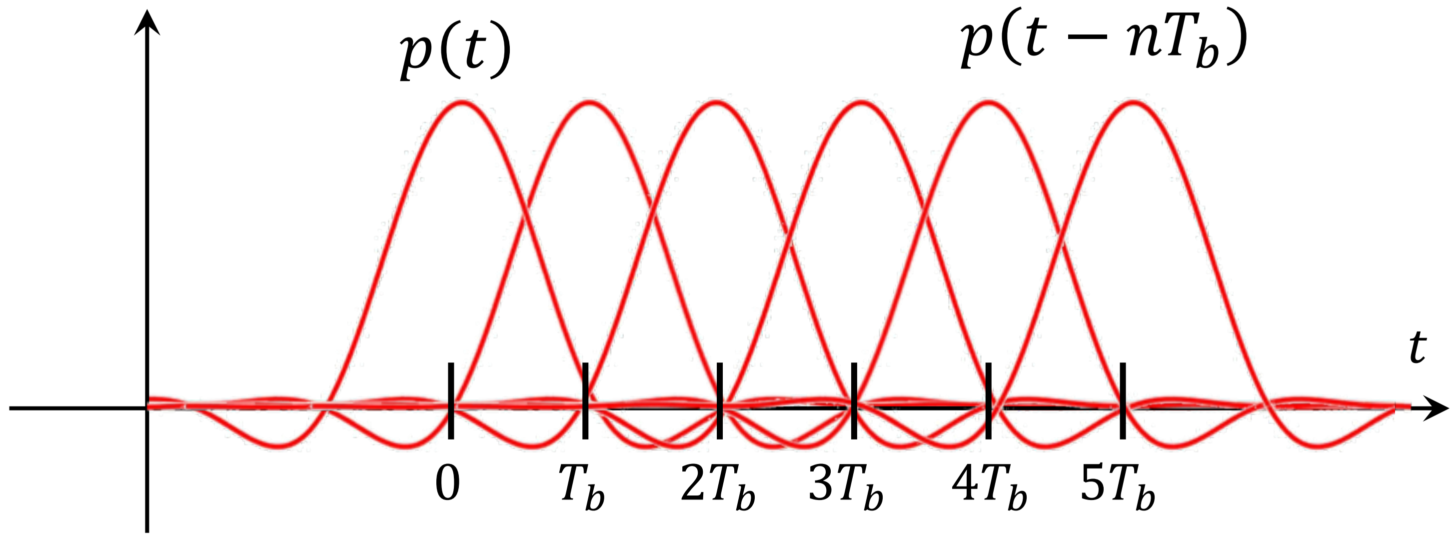
Symbol Timing Recovery



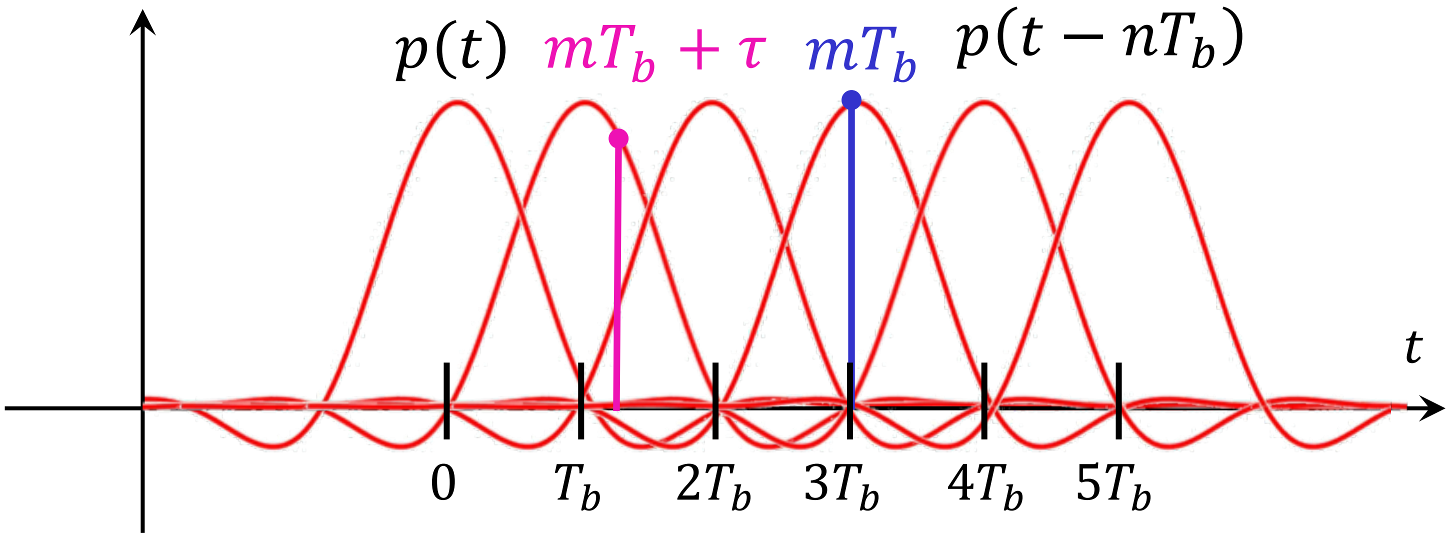
Symbol Timing Recovery



Symbol Timing Recovery



Symbol Timing Recovery



Must correct timing offset to avoid ISI!

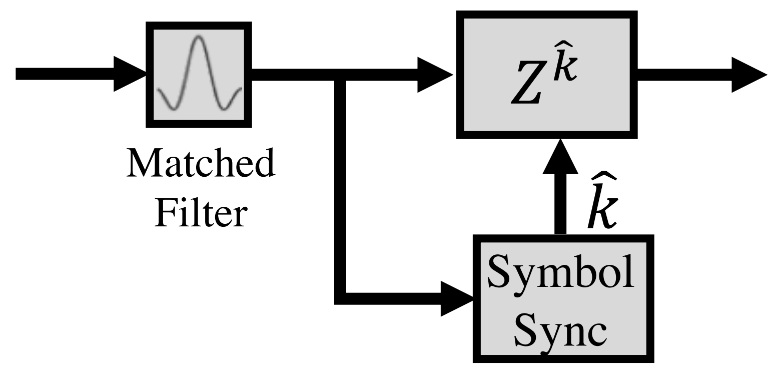
Power is maximum when $\tau = 0$

Symbol Timing Recovery

$s[n] \Rightarrow x(t) = s(t) * p_T(t) \Rightarrow y(t) = x(t) + v(t) \Rightarrow \tilde{y}(t) = y(t) * p_R(t)$

$$\tilde{y}(t) = \sum_n s[n]p(t - nT_b)$$

In practice, we have sampling offset:
Sample signal at $t = mT_b + \tau$



Compute: $J[k] = |\tilde{y}(t + k)|^2$

Find: $\hat{k} = \operatorname{argmax} J[k]$

Shift samples by :$\hat{k}$

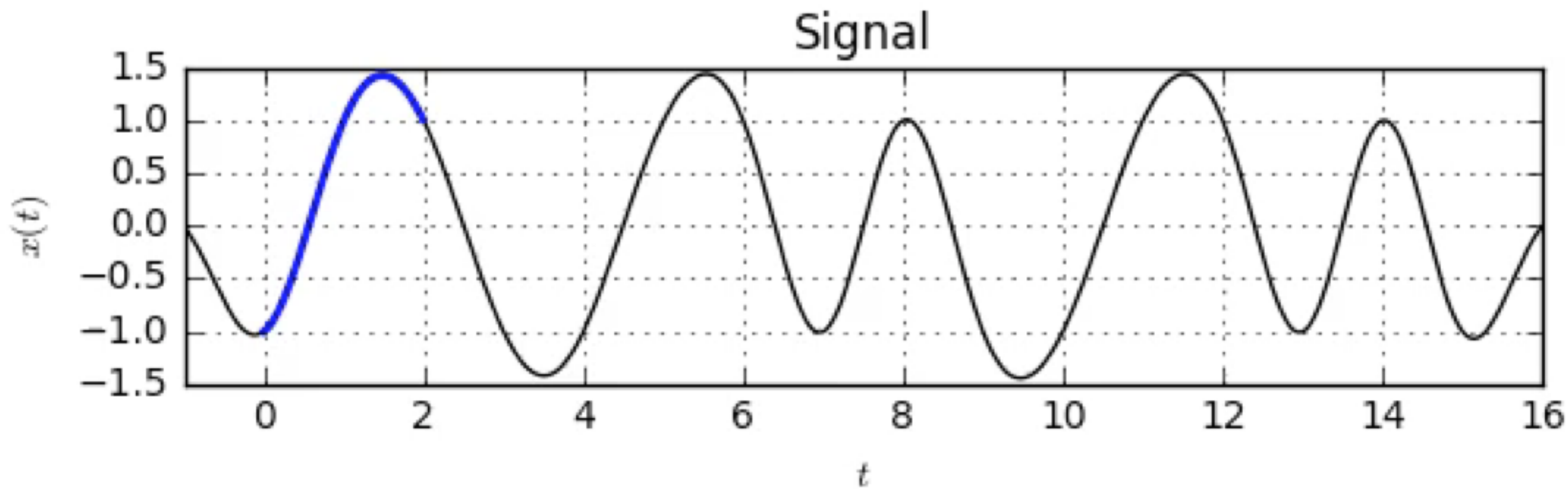
Eye Diagram

Qualitative tool to evaluate:

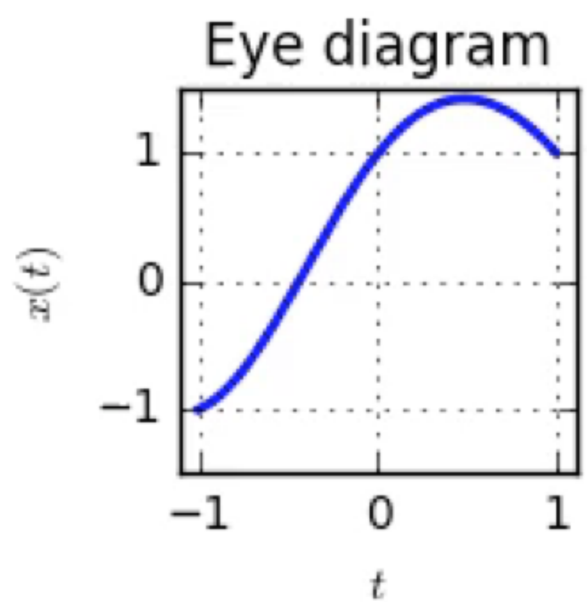
- Quality of Pulse
- ISI
- Distortion
- Noise

Generated by fragmenting the received signal $x(t)$ into overlapping fragments of length $2T_b$ and overlaying all the fragments.

Eye Diagram

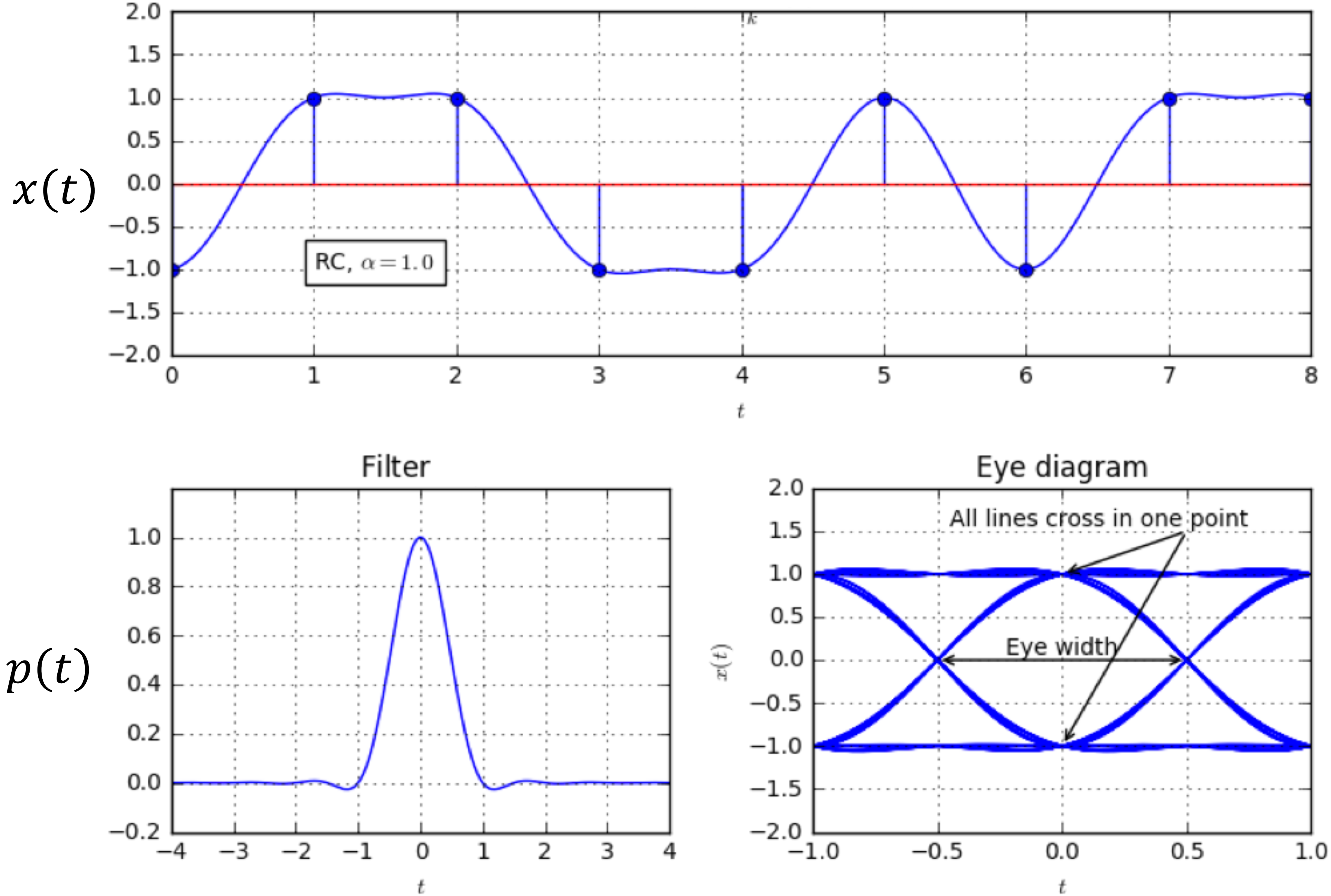


Signal
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
$x(t)$
0
2
4
6
8
10
12
14
16
t



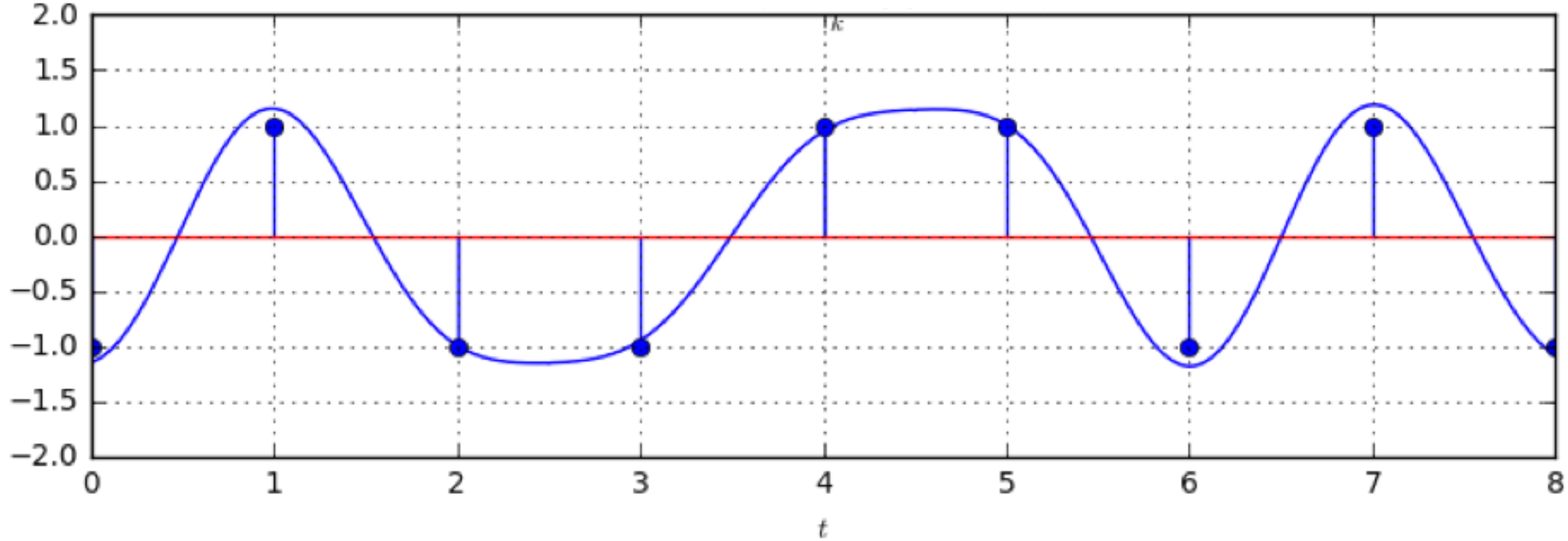
Eye diagram
1
0
−1
$x(t)$
−1
0
1
t

Eye Diagram: Raised Cosine $\alpha = 1$

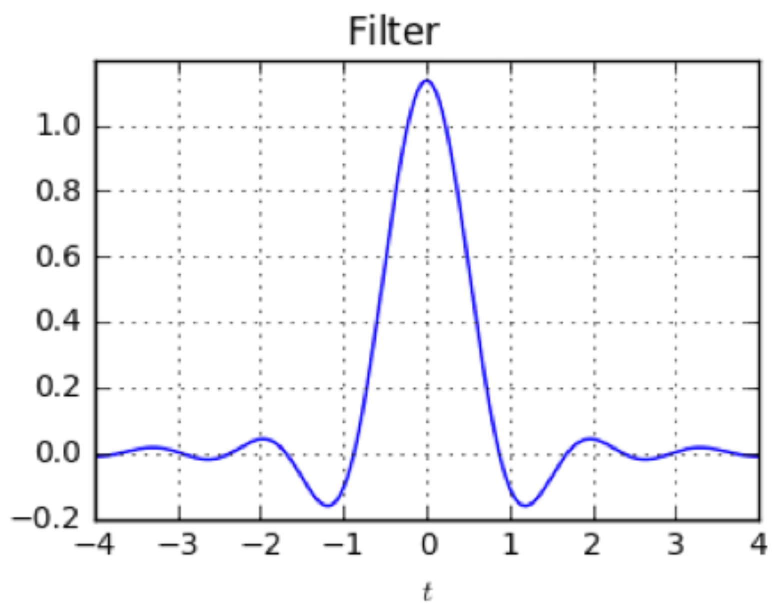


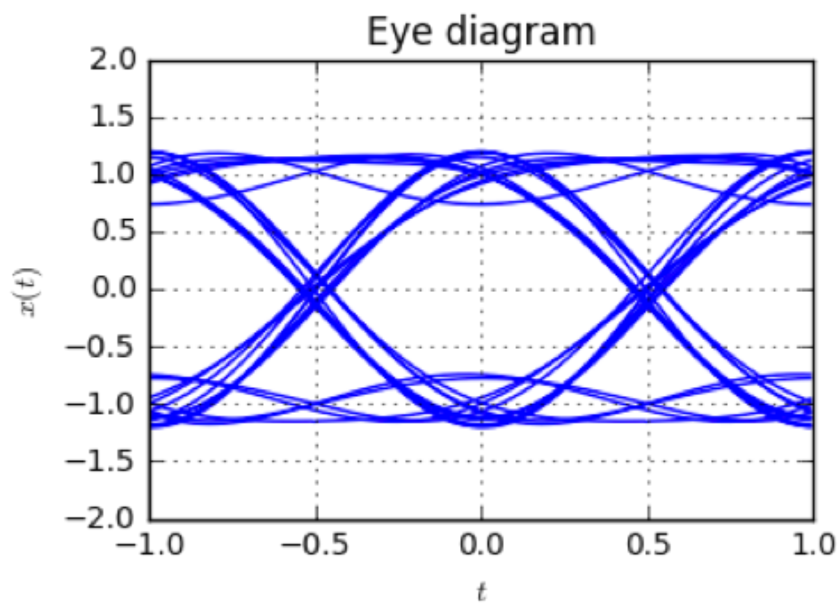
Eye Diagram: Raised Cosine $\alpha = 0.5$

$x(t)$



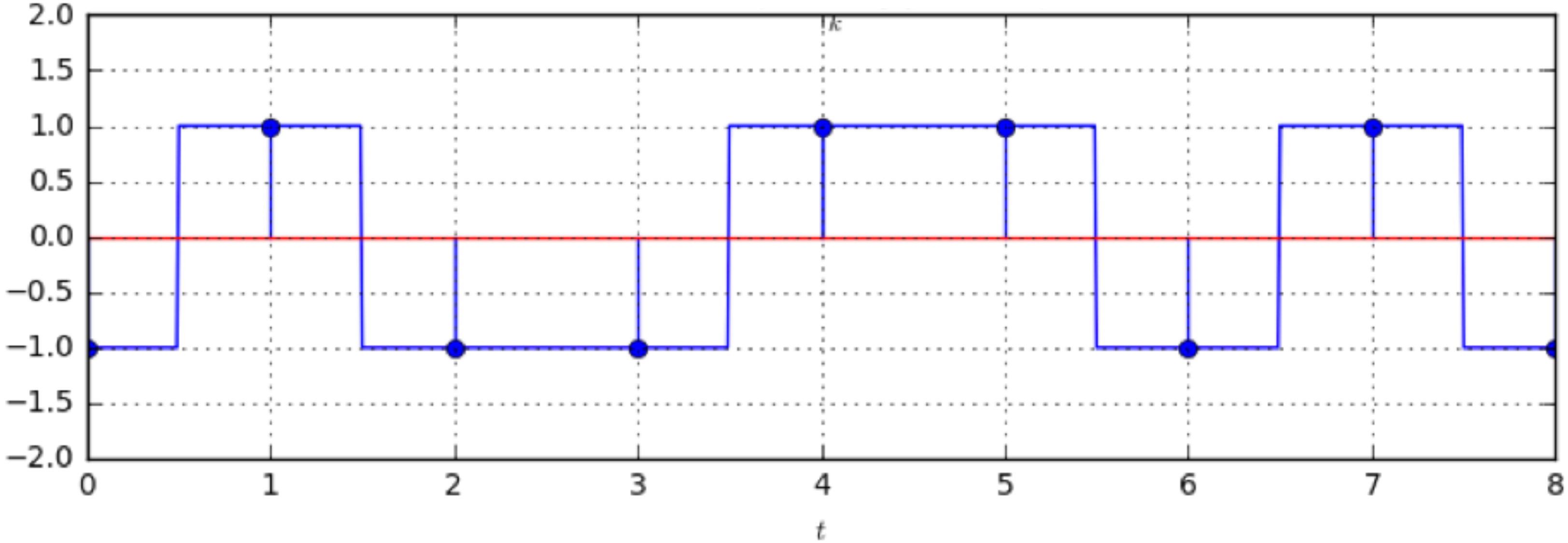
$p(t)$



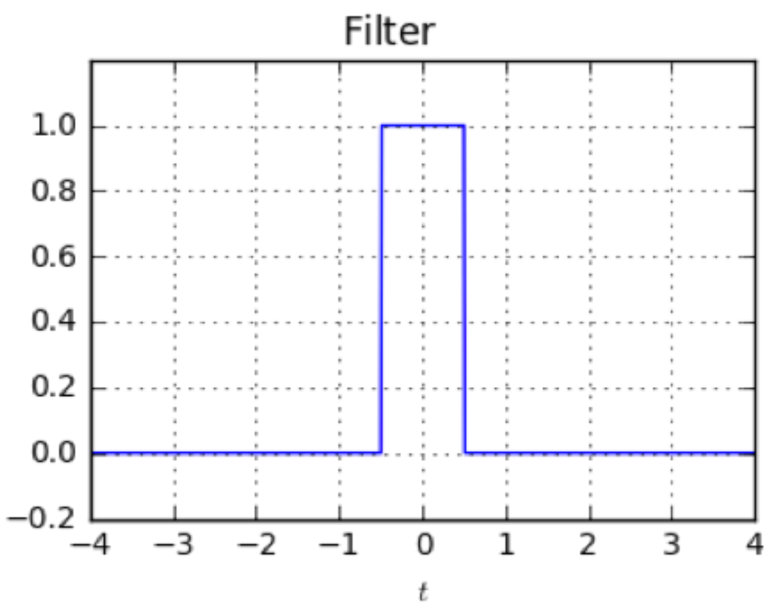


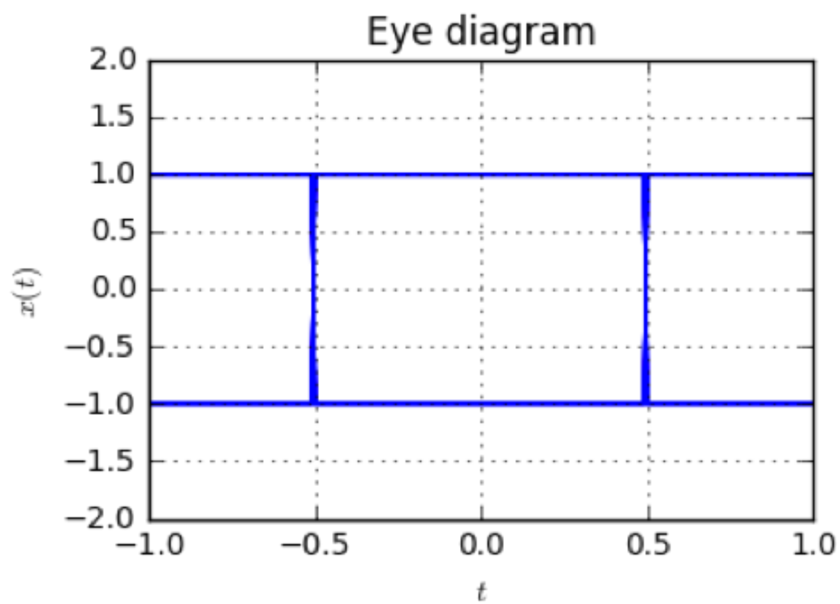
Eye Diagram: Rectangle

$x(t)$



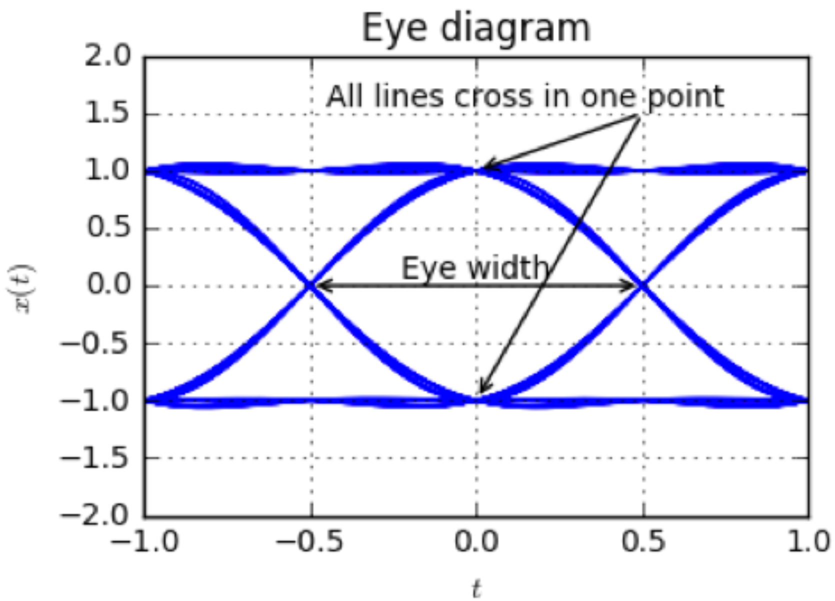
$p(t)$



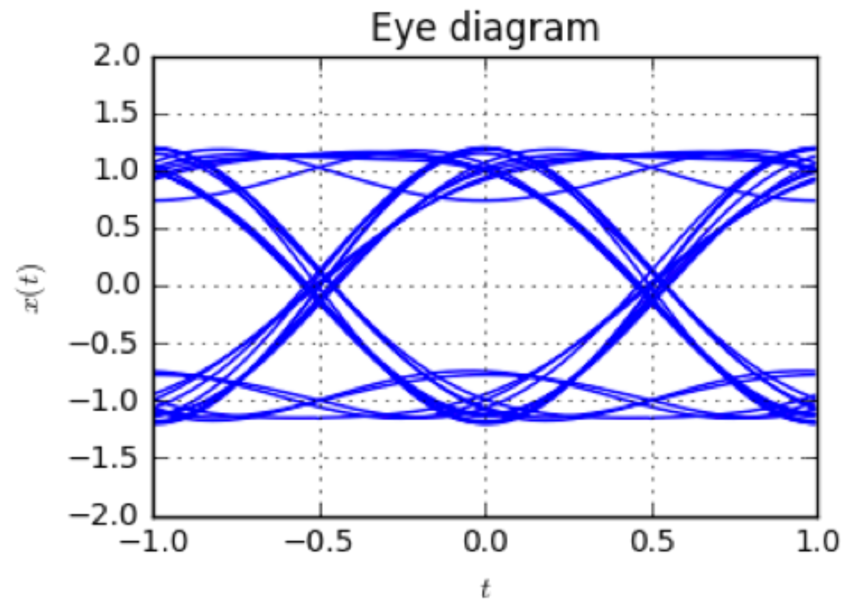


Eye Diagram

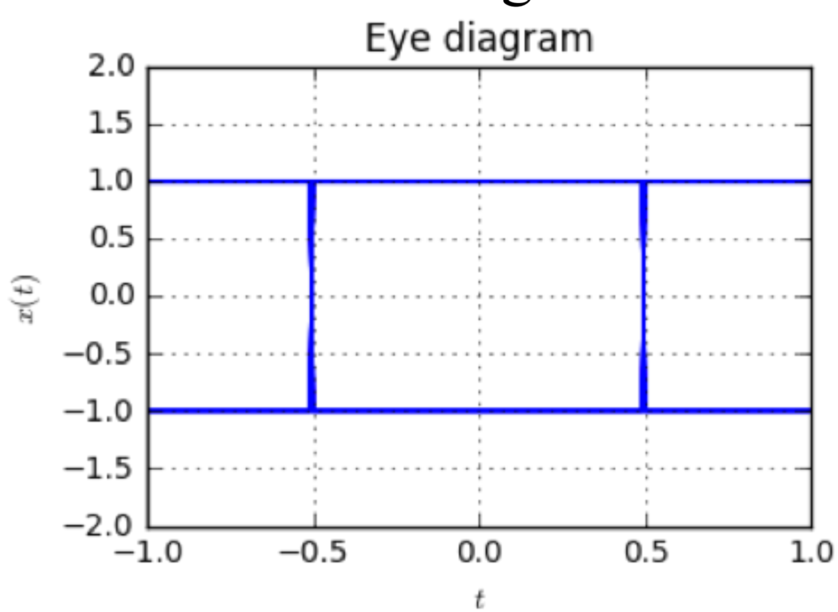
Raised Cosine $\alpha = 1$



Raised Cosine $\alpha = 0.5$

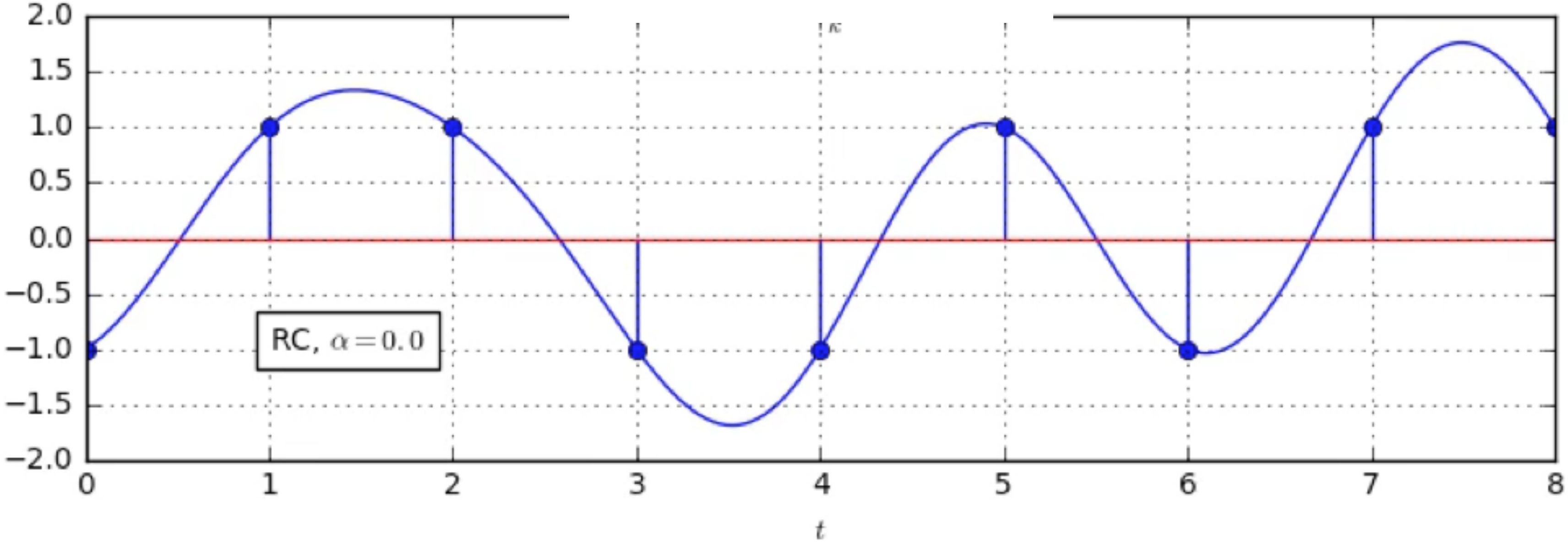


Rectangular

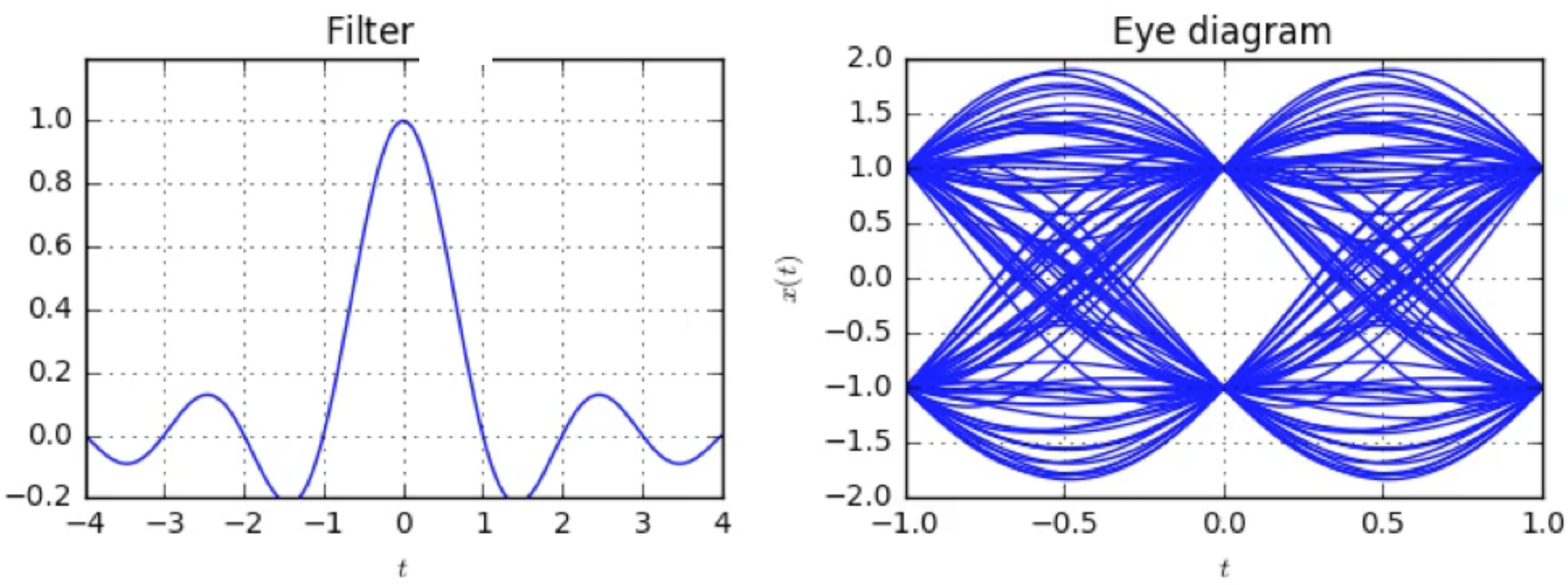


Eye Diagram

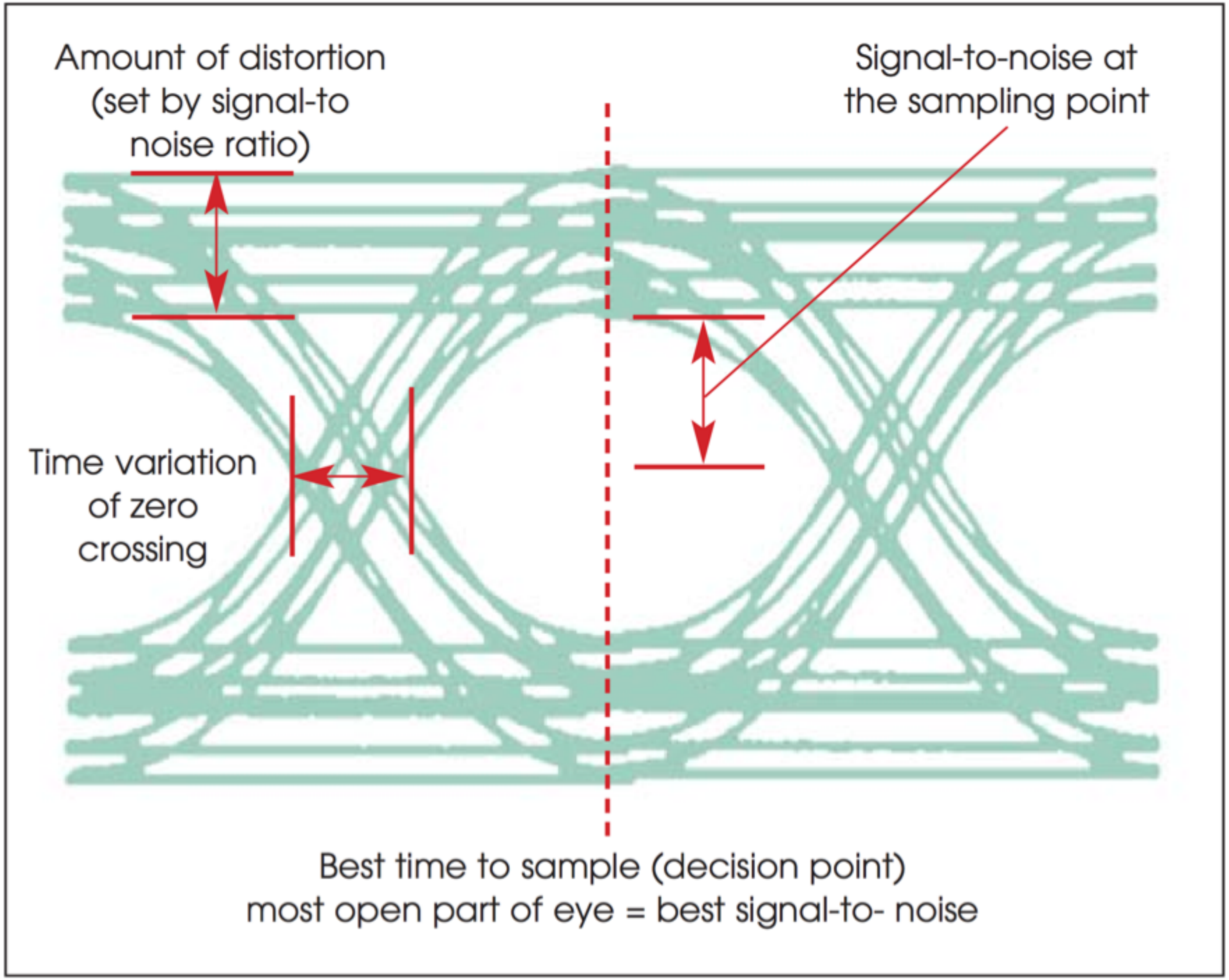
$x(t)$



$p(t)$

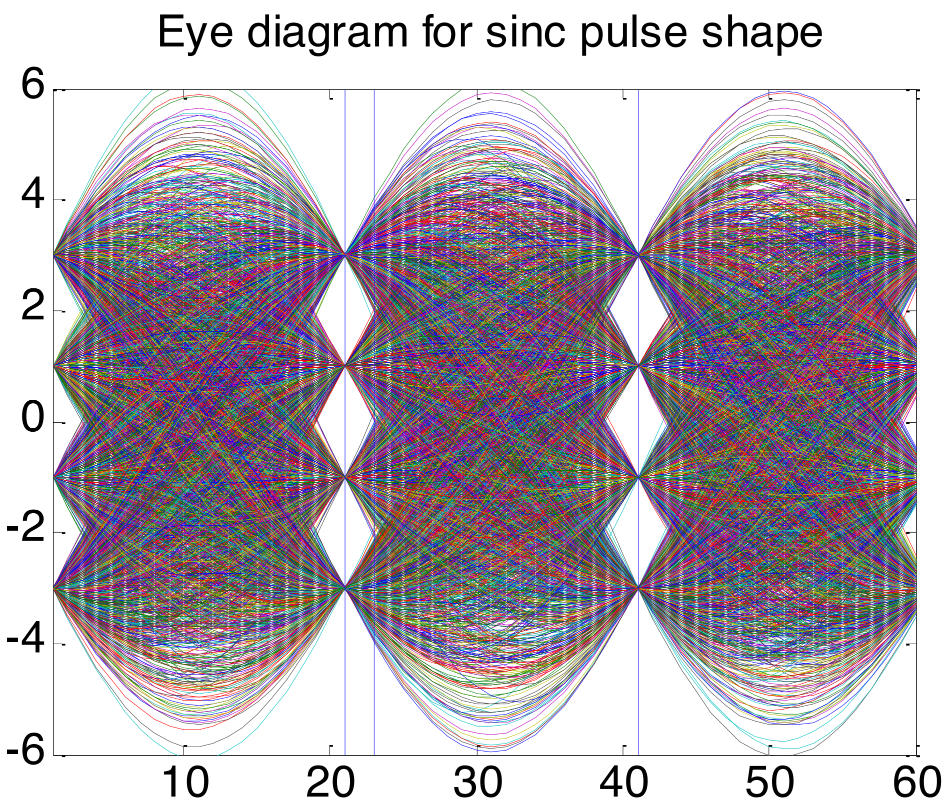


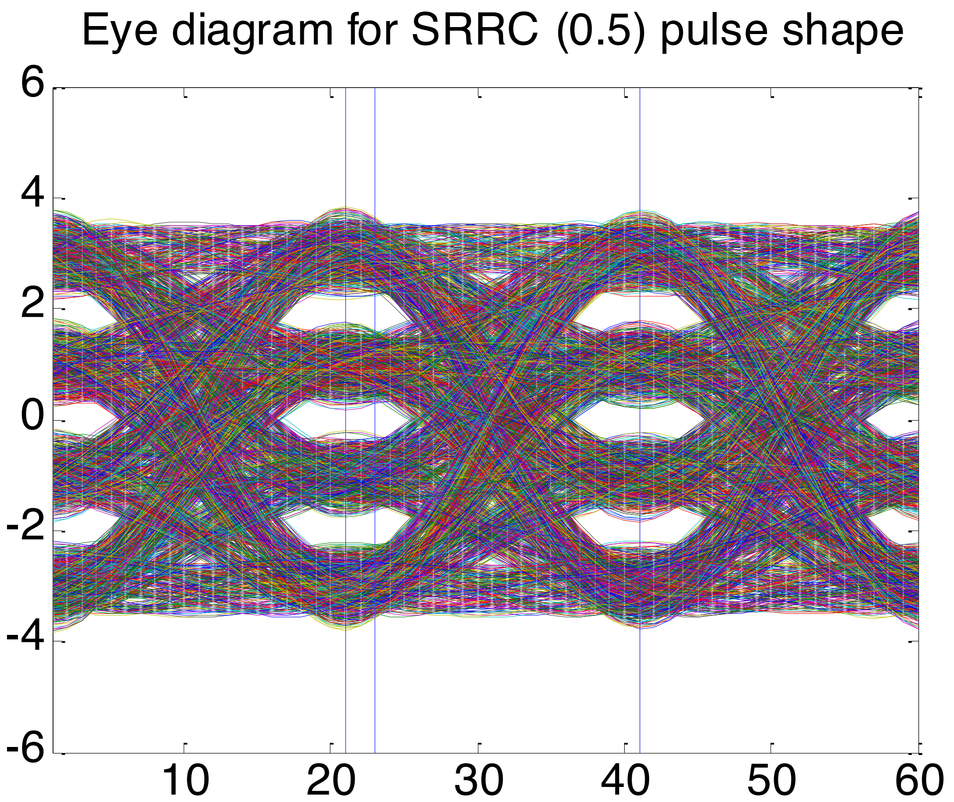
Eye Diagram



Amount of distortion
(set by signal-to
noise ratio)
Signal-to-noise at
the sampling point
Time variation
of zero
crossing
Best time to sample (decision point)
most open part of eye = best signal-to- noise

Eye Diagram: 4PAM





Definitions & Variables

- $b[n]$: Bit steam
- $s[n]$: Symbols
- n: Symbol index at TX
- m: Sampling index at RX
- $x(t)$: Baseband Signal
- $p(t)$: Pulse Shape
- $P(f)$: Frequency Spectrum of pulse
- $r[n]$: Sampled values at the receiver.
- T_b: Symbol Time
- R_b: Symbol Rate
- $\Pi(\quad)$: Rectangle Function
- $\text{sinc}(\quad)$: Sinc Function
- α: Roll-off Factor of Raised Cosine
- $(\quad)^*$: Complex Conjugate
- $|\quad|$: Magnitude
- $E[\quad]$: Expectation
- $\mathcal{F}\{\quad\}$: Fourier Transform
- $\mathcal{F}^{-1}\{\quad\}$: Inverse Fourier Transform
- $p_T(t)$: Transmitter Pulse
- $p_R(t)$: Receiver Pulse
- $v(t)$: Additive Gaussian Noise
- $P_T(f)$: Spectrum of TX pulse
- $P_R(f)$: Spectrum of RX pulse
- $P_{rc}(f)$: Spectrum of raised cosine pulse
- $P_{srrc}(f)$: Spectrum of square root raised cosine.
- $V(f)$: Frequency Spectrum of Noise
- N_0: Energy of noise spectrum
- $y(t)$: Received Signal
- $\tilde{y}(t)$: Received Signal after Matched Filter
- $\tilde{y}_s(t)$: Signal component of $\tilde{y}(t)$
- $\tilde{y}_v(t)$: Noise component of $\tilde{y}(t)$
- $\tilde{Y}_s(f)$: Frequency Spectrum of $\tilde{y}_s(t)$
- τ: Sampling offset
- $J[k]$: Magnitude of pulse samples
- $\hat{k}$: Sample index of pulse peak
- $Z^{\hat{k}}$: Z-transform representing delay of $\hat{k}$ samples