

ECE 457- Microwave Devices and Circuits

Fall 2026

Homework #3 - Transmission lines

Solutions

Pozar 2.7)

Using KVL:

$$-v(z) + R \frac{\Delta z}{2} i(z) + L \frac{\Delta z}{2} \frac{\partial i(z)}{\partial t} + R \frac{\Delta z}{2} i(z + \Delta z) + L \frac{\Delta z}{2} \frac{\partial i(z + \Delta z)}{\partial t} + v(z + \Delta z) = 0$$

Dividing by Δz and letting $\Delta z \rightarrow 0$:

$$\frac{\partial v(z)}{\partial z} = -R i(z) - L \frac{\partial i(z)}{\partial z}$$

Using KCL:

$$i(z) - \Delta z \left[G + C \frac{\partial}{\partial t} \right] \left[v(z) - \frac{\Delta z}{2} \left(R + L \frac{\partial}{\partial t} \right) i(z) \right] - i(z + \Delta z) = 0$$

Dividing by Δz and letting $\Delta z \rightarrow 0$:

$$\frac{\partial i(z)}{\partial z} = -G v(z) - C \frac{\partial v(z)}{\partial z}$$

2. A lossless tline with $Z_0 = 100\ \Omega$ is terminated with a load impedance $Z_L = 110 = j70\ \Omega$.

(a) First we need to determine the input impedance to the line. This can be found by the expression,

$$Z_{\text{in}} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_L + jZ_0 \tan(\beta l)}$$

where $\beta l = 2\pi \times 0.692$ radians. Inserting all the relevant quantities, we find $Z_L = 96 - j66\ \Omega$ or $z_{\text{in}} = 0.96 - j0.66$. The admittance looking into the loaded line is thus,

$$Y_{\text{in}} = \frac{1}{Z_{\text{in}}} = \frac{1}{96 - j66}\ \text{S} = 7.1 + j4.9\ \text{mS} \quad \rightarrow \quad y_{\text{in}} = 0.71 + j0.49$$

Thus, to make the input impedance (or, equivalently, the input admittance) purely real, we need to add a shunt element with normalized susceptance (B) or normalized reactance (X) of,

$$b = -0.49 \text{ or } B = -4.9\ \text{mS}, \quad x = -\frac{1}{b} = 2.04 \text{ or } X = 204\ \Omega$$

across the input terminals of the line.

(b) If this is done, the input impedance becomes,

$$Z_{\text{in}} = \frac{1}{7.1 \times 10^{-3}}\ \Omega = 140.8\ \Omega$$

(c) This required reactance (or susceptance) can be realized in a number of different ways using transmission lines. A simple method is to use a shunt short-circuited transmission line which has reactance,

$$X_{\text{SC}} = Z_{\text{SC}} \tan(\theta_{\text{SC}})$$

where Z_{SC} is the characteristic impedance of the line, and θ_{SC} is its electrical length. Choosing $Z_{\text{SC}} = 100\ \Omega$, we can tune out the susceptance with an open-circuited line that has electrical length of

$$2.04 = \tan(\theta_{\text{SC}}) \quad \Rightarrow \quad \theta_{\text{SC}} = 63.9^\circ$$

3. (a) For an ideal transformer with turns ratio of $1 : n$, the current voltage relations are,

$$V_2 = nV_1, \quad I_1 = nI_2$$

$$V_1 = V_0^+(1 + \Gamma) \quad \text{and} \quad I_1 = \frac{V_0^+}{Z_{01}}(1 - \Gamma) \quad (1)$$

where V_0^+ is the amplitude of the incident voltage wave at port 1. Similarly, at port 2, we can write the voltage and current as,

$$V_2 = V_t^+ = nV_1 \quad \text{and} \quad I_2 = \frac{V_t^+}{Z_{02}} = \frac{I_1}{n} \quad (2)$$

where V_t^+ is the transmitted voltage wave at port 2. Combining (1) and (2), we have

$$V_0^+(1 + \Gamma) = \frac{V_t^+}{n} \quad \text{and} \quad \frac{V_0^+}{Z_{01}}(1 - \Gamma) = n \frac{V_t^+}{Z_{02}}$$

Taking the ratio of these two expression gives us

$$Z_{01} \frac{1 + \Gamma}{1 - \Gamma} = \frac{Z_{02}}{n^2} \quad \text{or} \quad \Gamma = \frac{z - n^2}{z + n^2}$$

where $z \equiv Z_{02}/Z_{01}$. For the transmission coefficient, τ , we have

$$\tau \equiv \frac{V_t^+}{V_0^+} = n(1 + \Gamma) \quad \Rightarrow \quad \tau = \frac{2nz}{z + n^2}$$

(b) Let's begin by finding the impedance on the “ n ”-side of the transformer. For the 90° long tline we have (using the expression for a quarter-wave transformer),

$$Z_{\text{in}} = \frac{Z_0^2}{Z_L} = \frac{(100 \Omega)^2}{200 \Omega} = 50 \Omega$$

For the 112.5° long tline the reflection coefficient *at the load* is,

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{j200}{200(1 + j)} = \frac{1}{2} + j\frac{1}{2} = \frac{1}{\sqrt{2}} e^{j45^\circ}$$

At the input of the 112.5° line, the reflection coefficient, Γ_{in} , and impedance (Z_{in}) are,

$$\Gamma_{\text{in}} = \Gamma_L \exp(-j2\beta l) = \frac{1}{\sqrt{2}} \exp(j[45^\circ - 225^\circ]) = \frac{1}{\sqrt{2}} e^{j180^\circ} = -\frac{1}{\sqrt{2}}, \quad \text{and}$$

$$Z_{\text{in}} = Z_0 \frac{1 + \Gamma_{\text{in}}}{1 - \Gamma_{\text{in}}} = 100 \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \Omega = 17.16 \Omega$$

Thus, the total load impedance on the “ n ”-side of the transformer is,

$$Z_{\text{total}} = 50 \Omega \parallel 17.16 \Omega = 12.77 \Omega$$

Thus, the normalized impedance z is

$$z = \frac{12.77 \Omega}{50 \Omega} = 0.255$$

and for $\Gamma = 0$, we require

$$z = n^2 = 0.255 \quad \Rightarrow \quad n = 0.505$$

4. (a) We found the reflection coefficient for the series lumped elements in class (lecture 8, example 2):

$$\Gamma = \frac{z}{z + 2}$$

Thus, for the series resistor ($Z = R$) and inductor ($Z = j\omega L$), we have

$$\Gamma_R = \frac{R}{R + 2Z_0} \quad \Gamma_L = \frac{j\omega L}{2Z_0 + j\omega L}$$

For a shunt lumped element, it is easier to work in terms of admittances,

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \frac{\frac{1}{Z_L Z_0}}{\frac{1}{Z_L Z_0}} = \frac{Y_0 - Y_L}{Y_0 + Y_L}$$

where $Y_0 \equiv 1/Z_0$, and $Y_L = 1/Z_L$. The effective “load” admittance for the shunt lumped element is, $Y_L = Y + Y_0$, and thus,

$$\Gamma = -\frac{Y}{Y + 2Y_0} = -\frac{y}{y + 2}$$

Thus, for the shunt conductance ($Y = G$) and capacitance ($Y = j\omega C$), we have

$$\Gamma_G = -\frac{G}{G + 2Y_0} \quad \Gamma_C = \frac{j\omega C}{2Y_0 + j\omega C}$$

(b) For $Z \ll Z_0$ and $Y \ll Y_0$, the above expressions simplify to

$$\begin{aligned}\Gamma_R &\simeq \frac{R}{2Z_0}, \quad R \ll Z_0 & \Gamma_L &\simeq j\frac{\omega L}{2Z_0}, \quad \omega L \ll Z_0 \\ \Gamma_G &\simeq -\frac{G}{2Y_0}, \quad G \ll Y_0 & \Gamma_C &\simeq -j\frac{\omega C}{2Y_0}, \quad \omega C \ll Y_0\end{aligned}$$

(c) If an pulse is incident on each of these lumped elements, then the reflected pulse will look like the sketches below. For the inductor and capacitor, you should recall that the factor of $j\omega$ in the frequency domain corresponds to differentiation in the time-domain.

From each of these sketches, it is evident that we can deduce the nature of the discontinuity from a simple examination of the shape and size of the reflected pulse. This technique can be used to find “faults” in buried cables and is called “time-domain reflectometry”.

