

ECE 457- Microwave Devices and Circuits

Fall 2026

Homework #2 Solution – Plane Waves

Due:

1. (a) Consider the phasor $\vec{H}$ -field given by,

$$\vec{H} = \frac{1}{120\pi}(\hat{x} - 2\hat{y}) \exp(-jk_0 z)$$

From Amperère's Law, $\vec{\nabla} \times \vec{H} = j\omega\epsilon\vec{E}$, so that

$$j\omega\epsilon\vec{E} = \hat{y}\frac{\partial H_x}{\partial z} - \hat{x}\frac{\partial H_y}{\partial z} = -j\frac{k_0}{120\pi}(2\hat{x} + \hat{y}) \exp(-jk_0 z) \quad \text{or,}$$

$$\vec{E} = -(2\hat{x} + \hat{y}) \exp(-jk_0 z)$$

where we have made use of the relation,

$$\frac{k_0}{\omega\epsilon} = \sqrt{\frac{\mu_0}{\epsilon}} \approx 120\pi$$

The real part of the Poynting vector is give by,

$$\begin{aligned} P &= \text{Re}\{\vec{S}\} = \text{Re}\{\vec{E} \times \vec{H}^*\} = \frac{1}{120\pi}(\hat{x} - 2\hat{y}) \times (2\hat{x} + \hat{y}) \\ &= \hat{z}(E_x H_y^* - E_y H_x^*) \end{aligned}$$

$$\text{Re}\{\vec{E} \times \vec{H}^*\} = \frac{1}{24\pi}\hat{z}$$

- (b) Now consider the phasor $\vec{E}$ -field given by,

$$\vec{E} = \hat{x}(2 \cos k_0 y + e^{jk_0 y}) + \hat{z} 2 \sin k_0 y$$

From Faraday's Law, $\vec{\nabla} \times \underline{\vec{E}} = -j\omega\mu\underline{\vec{H}}$, we have

$$-j\omega\mu\underline{\vec{H}} = \hat{x}\frac{\partial \underline{E}_z}{\partial y} - \hat{z}\frac{\partial \underline{E}_x}{\partial y} = \hat{x}k_0(e^{-jk_0y} + e^{jk_0y}) - \hat{z}k_0(-je^{-jk_0y} + 2je^{jk_0y})$$

$$\underline{\vec{H}} = \hat{x}j2\frac{k_0}{\omega\mu}\cos(k_0y) + \hat{z}\frac{k_0}{\omega\mu}(e^{jk_0y} + j2\sin(k_0y))$$

Note that $\omega\mu/k_0 = \sqrt{\mu/\epsilon} = \eta$.

$$\begin{aligned}\underline{\vec{E}} \times \underline{\vec{H}}^* &= \hat{y}\left(\underline{E}_z H_x^* - \underline{E}_x H_z^*\right) \\ &= -\hat{y}\left(\frac{k_0}{\omega\mu} + \frac{2k_0}{\omega\mu}e^{-jk_0y}\cos(k_0y) - j\frac{2k_0}{\omega\mu}e^{jk_0y}\sin(k_0y)\right) \\ &= \frac{\hat{y}}{\eta}\left(j2e^{jk_0y}\sin(k_0y) - 2e^{-jk_0y}\cos(k_0y) - 1\right)\end{aligned}$$

$$\text{Re}\left\{\underline{\vec{E}} \times \underline{\vec{H}}^*\right\} = -\frac{\hat{y}}{\eta}\left(2\sin^2(k_0y) + 2\cos^2(k_0y) + 1\right) = -\frac{3}{\eta}\hat{y}$$

2. Consider two plane waves travelling in opposite directions:

$$\underline{\vec{E}} = \hat{x}Ae^{-jk_0z} + \hat{x}Be^{jk_0z}$$

(a) The magnitude of the electric field is,

$$|\underline{E}| = \left|Ae^{-jk_0z}\left(1 + \frac{B}{A}e^{j2k_0z}\right)\right| = A\sqrt{\left(1 + \frac{B}{A}\cos(2k_0z)\right)^2 + \left(\frac{B}{A}\sin(2k_0z)\right)^2}$$

or,

$$|E| = A\sqrt{1 + \left(\frac{B}{A}\right)^2 + 2\frac{B}{A}\cos(2k_0z)}$$

Clearly $|E|$ is a function of position, z . The maximum value of $|E|$ occurs when $k_0 z = n\pi$, where n is an integer:

$$|E|_{\max} = A\sqrt{\left(1 + \frac{B}{A}\right)^2} = A + B$$

The minimum magnitude occurs when $k_0 z = m\pi/2$, where m is an *odd* integer:

$$|E|_{\min} = A\sqrt{\left(1 - \frac{B}{A}\right)^2} = |A - B|$$

Because maxima occur when $z = n\pi/k_0 = n\lambda/2$ and minima occur at $z = m\pi/2k_0 = m\lambda/4$ (m is odd), the distance d between adjacent minima and maxima is,

$$d = \frac{\lambda}{4} \quad \text{a quarter wavelength}$$

(b) From Faraday's Law, the magnetic field is

$$\vec{\nabla} \times \vec{E} = \hat{y} \frac{\partial E_x}{\partial z} = -jk_0 \hat{y} (Ae^{-jk_0 z} + Be^{jk_0 z}) = -j\omega\mu \vec{H}$$

Thus,

$$\vec{H} = \left(\frac{A}{\eta} e^{-jk_0 z} - \frac{B}{\eta} e^{jk_0 z} \right) \hat{y}, \quad \text{as expected}$$

(c) The wave impedance is the ratio of the electric field phasor amplitude to the magnetic field phasor amplitude:

$$Z \equiv \frac{E}{H} = \frac{Ae^{-jk_0 z} + Be^{jk_0 z}}{\frac{A}{\eta} e^{-jk_0 z} - \frac{B}{\eta} e^{jk_0 z}} = \eta \frac{1 + \frac{B}{A} \exp(j2k_0 z)}{1 - \frac{A}{B} \exp(j2k_0 z)}$$

(d) If $A = -B$, then

$$|E|_{\max} = 2A, \quad \text{and} \quad |E|_{\min} = 0$$

$$Z(z) = \eta \frac{1 - \exp(j2k_0z)}{1 + \exp(j2k_0z)} = -j\eta \tan(k_0z)$$

∴ Standing Wave Ratio (SWR)

(a) for a uniform plane wave travelling in free space, $B = 0$ and

$$\vec{E} = \hat{z}Ae^{-jk_0z} \rightarrow |E|_{\max} = |E|_{\min} = A \rightarrow \text{SWR} = 1$$

(b) for a plane wave incident on a planar dielectric with refractive index n ,

$$\Gamma_0 = \frac{1-n}{1+n} \rightarrow |E|_{\max} = A(1 + |\Gamma_0|), \quad |E|_{\min} = A(1 - |\Gamma_0|)$$

Thus,

$$\text{SWR} = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|} = n$$

(c) Suppose $\text{SWR} = 3$ and the distance to the first $\vec{E}$ -field minimum is $5\lambda_0/16$:

$$|\Gamma_0| = \frac{\text{SWR} - 1}{\text{SWR} + 1} = \frac{1}{2}$$

and the minimum occurs where,

$$\exp[j(\phi + 2k_0z)] = -1$$

Note that if the interface lies at $z = 0$, then the value of z where the minimum electric field occurs will be negative. Thus, we have,

$$\phi - 2k_0\left(\frac{5\lambda_0}{16}\right) = -\pi \quad \Rightarrow \quad \phi = -\pi + \frac{4\pi}{\lambda_0}\left(\frac{5\lambda_0}{16}\right) = \frac{\pi}{4}$$

Thus,

$$\Gamma_0 = \frac{1}{2} \exp\left(j\frac{\pi}{4}\right), \quad \text{and}$$

$$n = \frac{1 - \Gamma_0}{1 + \Gamma_0} = 0.526\angle -43^\circ = 0.38 - j0.36$$

4. For a “good” conductor, $\sigma \gg \omega\epsilon$ and

$$Z = \sqrt{\frac{\omega\mu}{2\sigma}}(1 + j)$$

Thus,

$$\Gamma_0 = \frac{\sqrt{\frac{\omega\mu}{2\sigma}}(1 + j) - \sqrt{\frac{\mu_0}{\epsilon_0}}}{\sqrt{\frac{\omega\mu}{2\sigma}}(1 + j) + \sqrt{\frac{\mu_0}{\epsilon_0}}} = \frac{\sqrt{\frac{\omega^2\mu_0\epsilon_0}{2\omega\mu_0\sigma}}(1 + j) - 1}{\sqrt{\frac{\omega^2\mu_0\epsilon_0}{2\omega\mu_0\sigma}}(1 + j) + 1} = \frac{\frac{1}{2}k_0\delta_s(1 + j) - 1}{\frac{1}{2}k_0\delta_s(1 + j) + 1}$$

where we have used the relations,

$$\delta_s \equiv \sqrt{\frac{2}{\omega\mu_0\sigma}} \quad \text{and} \quad k_0^2 = \omega^2\mu_0\epsilon_0$$

For a good conductor, $k_0\delta_s \ll 1$ and from the Taylor series expansion, $(1 + x)^{-1} \simeq 1 - x$ for $|x| \ll 1$. This gives us,

$$\Gamma_0 \approx -\left(1 - \frac{1}{2}k_0\delta_s(1 + j)\right)^2 \approx -\left(1 - k_0\delta_s(1 + j)\right) \quad \Rightarrow \quad |\Gamma_0|^2 = (1 - k_0\delta_s)^2 + (k_0\delta_s)^2$$

Keeping only the largest terms, we have

$$|\Gamma_0|^2 \approx 1 - 2k_0\delta_s$$

Using the data tabulated in the text, we have:

(i) for copper, $\sigma = 5.813 \times 10^7 \text{ S/m} \rightarrow \delta_s = 660.6 \text{ nm}$,

$$1 - |\Gamma_0|^2 \text{ (dB)} = 10 \log(2k_0\delta_s) = -35.6 \text{ dB}$$

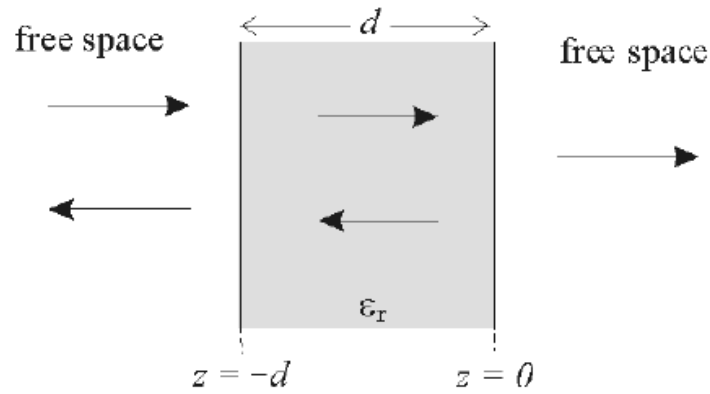
(ii) for brass, $\sigma = 2.564 \times 10^7 \text{ S/m} \rightarrow \delta_s = 994.5 \text{ nm}$,

$$1 - |\Gamma_0|^2 \text{ (dB)} = 10 \log(2k_0\delta_s) = -33.8 \text{ dB}$$

(iii) for stainless steel, $\sigma = 1.1 \times 10^6 \text{ S/m} \rightarrow \delta_s = 4.8 \mu\text{m}$,

$$1 - |\Gamma_0|^2 \text{ (dB)} = 10 \log(2k_0\delta_s) = -26.9 \text{ dB}$$

5. Pozar, 1.7; A plane wave is incident on a dielectric slab. Refer to the figure below:



(a) First consider the boundary at $z = 0$. To the right of the boundary, we have only $+z$ travelling waves:

$$\underline{E}_t = E_t e^{-jk_0 z}; \quad \underline{H}_t = \frac{E_t}{\eta_0} e^{-jk_0 z}$$

Inside the dielectric slab, we can write the total field as a superposition of two opposite directions,

$$\underline{E}_{\text{in}} = E_+ e^{-jkz} + E_- e^{jkz}; \quad \underline{H}_{\text{in}} = \frac{E_+}{\eta} e^{-jkz} - \frac{E_-}{\eta} e^{jkz}$$

Now, enforcing continuity of the tangential fields at $z = 0$, we have

$$\frac{\underline{E}_{\text{in}}}{\underline{H}_{\text{in}}} = \frac{\underline{E}_t}{\underline{H}_t} \Big|_{z=0} \Rightarrow \eta \frac{1 + \frac{E_-}{E_+}}{1 - \frac{E_-}{E_+}} = \eta_0$$

or,

$$\frac{E_-}{E_+} \equiv \Gamma'_0 = \frac{\eta_0 - \eta}{\eta_0 + \eta}$$

where $\eta = \sqrt{\mu_0/\epsilon}$ and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$.

At $z = -d$, the fields in the slab are,

$$\underline{E}_{\text{in}}(z = -d) = E_+ e^{jkd} + E_- e^{-jkd}; \quad \underline{H}_{\text{in}}(z = -d) = \frac{E_+}{\eta} e^{jkd} - \frac{E_-}{\eta} e^{-jkd}$$

The fields on the free-space side of the slab at $z = -d$ are,

$$\underline{E}_{\text{tot}} = \underline{E}_0 + \underline{E}_r; \quad \underline{H}_{\text{tot}} = \frac{\underline{E}_0}{\eta_0} - \frac{\underline{E}_r}{\eta_0}$$

Thus, applying Maxwell's boundary conditions at $z = -d$, we have

$$\frac{\underline{E}_{\text{in}}}{\underline{H}_{\text{in}}} = \frac{\underline{E}_{\text{tot}}}{\underline{H}_{\text{tot}}} \Big|_{z=-d} = \eta \frac{1 + \frac{E_-}{E_+} \exp(-j2kd)}{1 - \frac{E_-}{E_+} \exp(-j2kd)} = \eta_0 \frac{1 + \Gamma_0}{1 - \Gamma_0} \quad (*)$$

where $\Gamma_0 = \underline{E}_r/\underline{E}_0$. For $d = \lambda_0/4\sqrt{\epsilon_r}$, we have $2kd = \pi$ and thus,

$$\eta_0 \frac{1 + \Gamma_0}{1 - \Gamma_0} = \eta \frac{1 - \Gamma'_0}{1 + \Gamma'_0} = \eta \left(\frac{2\eta}{2\eta_0} \right) = \frac{\eta^2}{\eta_0}$$

As a result, we find that

$$\Gamma_0 = \frac{\eta^2 - \eta_0^2}{\eta^2 + \eta_0^2} = \frac{1 - \epsilon_r}{1 + \epsilon_r} = \frac{1 - n^2}{1 + n^2}$$

(b) If the thickness of the slab is $d = \lambda_0/2\sqrt{\epsilon_r}$, then $2kd = 2\pi$ and equation (*) above reduces to,

$$\eta \frac{1 + \frac{E_-}{E_+} \exp(-j2kd)}{1 - \frac{E_-}{E_+} \exp(-j2kd)} = \eta_0 \frac{1 + \Gamma_0}{1 - \Gamma_0} = \eta \frac{1 + \Gamma'_0}{1 - \Gamma'_0} = \eta \left(\frac{2\eta_0}{2\eta} \right) = \eta_0$$

or,

$$\Gamma_0 = \frac{\eta_0 - \eta_0}{\eta_0 + \eta_0} = 0$$

In other words, if the slab is a half-wavelength thick, then it is transparent to the incident wave (no reflection occurs)!