Blackbody Radiation, Gain and Broadening

ECE 455 Optical Electronics

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ECE Illinois

Introduction

In this section, we will learn how to do the following:

- Understand the properties of blackbody (thermal) light
- Predict the line width of atomic and molecular transitions
- Describe absorption, spontaneous emission and stimulated emission with rate equations

Blackbody Radiation

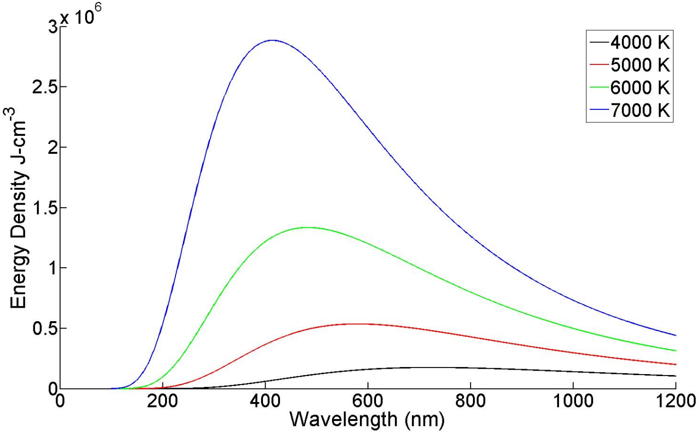
Blackbody radiation is emitted by objects because they are hot. The optical energy density for a blackbody radiator at temperature T between the frequencies of ν and $\nu + d\nu$ is

$$\rho(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{exp\left(\frac{h\nu}{k_B T}\right) - 1}d\nu \tag{1}$$

where ν is the frequency, h is Planck's constant, c is the speed of light, and k_B is Boltzmann's constant. The peak wavelength of this distribution can be found with the following equation

$$\lambda_{max} = \frac{2.898 \times 10^6 \text{ K}\cdot\text{nm}}{T} \tag{2}$$

Blackbody Radiation Picture



Example: Human Blackbody

Problem: The temperature of a healthy person is about 37° C. If a human could be considered an ideal blackbody, at what wavelength would the blackbody spectrum peak? What are the energy and photon densities of visible light emitted by people immediately above the surface of the skin? What are the energy and photon densities of photons in ±1% bandwidth around the peak wavelength?

Solution: The peak wavelength can be found with the aid of Equation

$$\lambda_p = \frac{2.898 \times 10^6 \text{ K}\cdot\text{nm}}{(37 + 273.15)\text{ K}} = 9344 \text{ nm} \tag{3}$$

Example: Human Blackbody

To find the energy density of visible light, numerically integrate Equation 1 over the appropriate limits:

$$\begin{aligned} \rho_{vis} &= \int_{\frac{c}{700\ \text{nm}}}^{\frac{c}{400\ \text{nm}}} \frac{8\pi h\nu^3}{c^3} \frac{1}{exp\left(\frac{h\nu}{k_B T}\right) - 1} d\nu \\ &= 5.44 \times 10^{-30}\ \text{J-cm}^{-3} \end{aligned} \tag{4}$$

To convert energy density to photon density, divide by the energy per photon, $h\nu$. Numerical integration is once again required

$$\begin{aligned} \bar{\rho}_{vis} &= \int_{\frac{c}{700\ \text{nm}}}^{\frac{c}{400\ \text{nm}}} \frac{8\pi \nu^2}{c^3} \frac{1}{exp\left(\frac{h\nu}{k_B T}\right) - 1} d\nu \\ &= 1.87 \times 10^{-11}\ \text{photons-cm}^{-3} \end{aligned} \tag{5}$$

Example: Human Blackbody

For a $\pm 1\%$ bandwidth around the peak, the energy density is

$$\rho_{IR} = \int_{\frac{0.99 \cdot c}{9344 \text{ nm}}}^{\frac{1.01 \cdot c}{9344 \text{ nm}}} \frac{8\pi h\nu^3}{c^3} \frac{1}{exp\left(\frac{h\nu}{k_B T}\right) - 1} d\nu \quad (6)$$

$$= 9.21 \times 10^{-8} \text{ J-cm}^{-3}$$

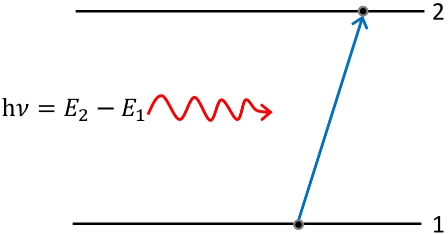
Because the bandwidth of this integration region is much narrower, there is no need to perform another numerical integration; we can simply divide the above answer by the energy per photon $\frac{hc}{\lambda_p} = 2.13 \times 10^{-20}$ J.

$$\bar{\rho}_{IR} = \frac{9.21 \times 10^{-8} \text{ J-cm}^{-3}}{2.13 \times 10^{-20} \text{ J/photon}} = 4.33 \times 10^{12} \text{ photons-cm}^{-3} \quad (7)$$

The lesson: If you are looking for somebody in the dark, don't use your eyes. Use an IR camera.

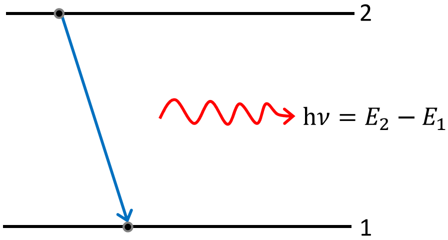
Three Atomic Processes: Absorption

- When a two-level system absorbs a photon, it is left in an excited state



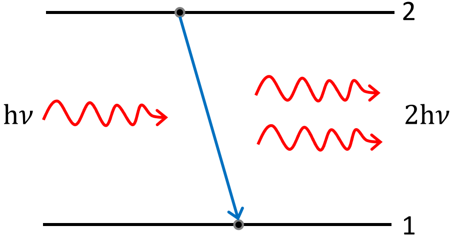
Three Atomic Processes: Spontaneous Emission

- Two-level system spontaneously decays from a higher energy level to a lower energy level and emits a photon.
- The decay rate is characterized by the **spontaneous emission lifetime** τ
- Emission is uniform in all 4π steradians
- Spontaneous emission can only be explained by QED



Three Atomic Processes: Stimulated Emission

- Photon interacts with two-level system in an excited state
- The system goes to a lower energy level and a photon with the corresponding energy is emitted
- Emitted photon has the same **direction, polarization, phase** and **energy** as the incident photon



Rates of the Three Atomic Processes

- Absorption

$$\left.\frac{dN_2}{dt}\right|_{Absorption} = +B_{12}\rho(\nu_{21})N_1 \tag{8}$$

- Spontaneous Emission

$$\left.\frac{dN_2}{dt}\right|_{SpontaneousEmission} = -A_{21}N_2 \tag{9}$$

- Stimulated Emission

$$\left.\frac{dN_2}{dt}\right|_{StimulatedEmission} = -B_{21}\rho(\nu_{21})N_2 \tag{10}$$

- Total population transfer

$$\frac{dN_2}{dt} = -A_{21}N_2 - B_{21}\rho(\nu_{21})N_2 + B_{12}\rho(\nu_{21})N_1 \tag{11}$$

Relationship Between Einstein Coefficients I

Assume atoms are in equilibrium with a blackbody radiation field. In steady state, $\frac{d}{dt}() = 0$. Solving Equation 14 for $\rho(\nu)$, we find:

$$\rho(\nu) = \frac{A_{21} N_2}{B_{12} N_1 - B_{21} N_2} \tag{12}$$

In thermal equilibrium, the populations of energy levels can be assumed to follow a Boltzmann distribution. Thus:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} exp\left(-\frac{h\nu_{21}}{k_B T}\right) \tag{13}$$

Relationship Between Einstein Coefficients II

Substitution of Equation 16 into 15 gives:

$$\rho(\nu_{21}) = \frac{A_{21}}{B_{21}} \frac{1}{\frac{B_{12} g_1}{B_{21} g_2} e^{h\nu_{21}/k_B T} - 1} \tag{14}$$

When the above equation is compared with the blackbody spectrum (Equation 1),

$$\rho(\nu) = \frac{8\pi h \nu^3}{c^3} \frac{1}{exp\left(\frac{h\nu}{k_B T}\right) - 1} \tag{1}$$

the following relationships are discovered:

$$B_{21} = \frac{g_1}{g_2} B_{12} \tag{15}$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} \tag{16}$$

Example: Relative Strength of Stimulated and Spontaneous Emission

Problem: Sodium atoms have contaminated the filament of an incandescent halogen lightbulb filament. When the bulb is operating at $T = 3100$ K, assume the sodium atoms and the optical field are in thermal equilibrium with the filament. Find the ratio of spontaneous emission to stimulated emission on the D_2 (589 nm) line.

Solution: The ratio of spontaneous emission to stimulated emission is:

$$\frac{R_{spont}}{R_{stim}} = \frac{A_{21} N_2}{B_{21} N_2 \rho(\nu_{21})} \tag{17}$$

Example: Relative Strength of Stimulated and Spontaneous Emission II

By substituting Equations 1 and 21 into the previous expression, we find:

$$\frac{R_{spont}}{R_{stim}} = \frac{8\pi h\nu^3}{c^3}\left[\frac{8\pi h\nu^3}{c^3}\frac{1}{exp\left(\frac{h\nu}{k_B T}\right)-1}\right]^{-1} \quad (18)$$

$$= exp\left(\frac{hc}{\lambda k_B T}\right)-1 \quad (19)$$

$$= 2652$$

If the atoms were further away from the light bulb, where the light is less intense, this ratio would increase even more.

The Lesson: In thermal light, spontaneous emission is much stronger than stimulated emission.

Example: Relative Strength of Stimulated and Spontaneous Emission

Problem: Suppose the atoms are instead excited by a sodium laser with 1 W average power and a mode area of 4 mm^2. Find the ratio of stimulated to spontaneous emission.

Solution: First we must estimate the intensity of the

$$I = \frac{1 \text{ W}}{4 \text{ mm}^2} = 250000 \text{ W/m}^2 \tag{20}$$

We'll see later that intensity can be converted to energy density by dividing by c

$$\rho_{21} = I/c = 8.33 \times 10^{-4} \text{ J/m}^3 \tag{21}$$

Example: Relative Strength of Stimulated and Spontaneous Emission

Writing an equation similar to Equation 21

$$\frac{R_{spont}}{R_{stim}} = \frac{A_{21}}{B_{21}\rho(\nu_{21})} \tag{22}$$

$$= \frac{8\pi h\nu^3}{c^3}\frac{1}{\rho_{21}\cdot g(\nu_{21})} \tag{23}$$

$$= \frac{8\pi h}{\lambda^3}\frac{1}{\rho_{21}}\frac{\Delta\nu_D}{2}\sqrt{\frac{\pi}{ln2}} \tag{24}$$

$$= 0.1613 \tag{25}$$

Note $\rho(\nu_{21}) = g(\nu_{21})\rho_{21}$. The $g(\nu_{21})$ term will be discussed later.

The Lesson: Blackbody radiation may have a higher power density, but most of the energy is not "in-band" and does not interact with the transition.

Introduction to Lineshape

- Atomic and molecular transition lines have a finite bandwidth; They are not perfectly sharp.
- Various physical processes broaden the spectrum lines.
- **Homogeneous** line broadening occurs when all emitters are equally affected by broadening mechanisms.
- **Inhomogeneous** line broadening occurs when all emitters are unequally affected by broadening mechanisms.
- Line broadening reduces the effective gain because not all atoms are capable of interacting with the radiation field.
- Macroscopic lasers will operate at a wavelength near the peak of the gain unless deliberately tuned otherwise.

The Lineshape Function

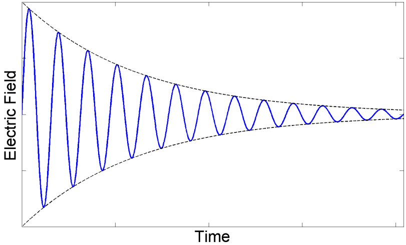
If N_2 atoms are in an upper energy level, then the number capable of emitting a photon within a $d\nu$ wide band centered around the frequency ν is:

$$N(\nu)d\nu = g(\nu)N_2 d\nu \tag{26}$$

If this equation is integrated over all frequencies, we must get N_2. Thus $g(\nu)$ must be a probability distribution. In other words:

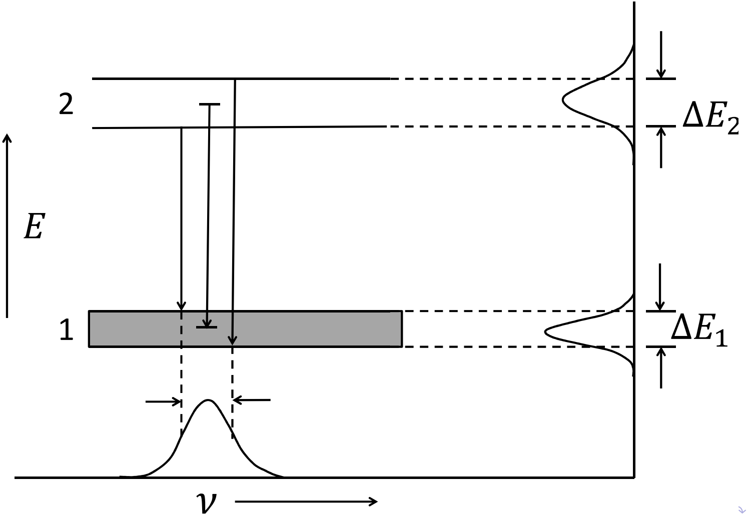
$$\int_0^\infty g(\nu)d\nu = 1 \tag{27}$$

Lifetime Broadening: Phenomenological View



- Broadening of lines can be seen from a Fourier transform perspective
  - $E(t) = E_0 u(t) e^{-\kappa_0 t/2} e^{\imath \omega_0 t} \Leftrightarrow E(\omega) = \frac{-E_0}{\imath(\omega - \omega_0) + \kappa_0/2}$
- Broadening of states can be viewed as a manifestation of time-energy uncertainty
  - $\Delta E \Delta t \geq \frac{\hbar}{2}$
- Can more rigorously be derived with time-dependent quantum perturbation theory

Lifetime Broadening: Blurred Energy Levels



Lifetime Broadening: Spectral Shape

Lifetime broadening results in a Lorentzian profile

$$g(\nu) = \frac{1}{2\pi} \frac{\Delta\nu}{(\nu - \nu_0)^2 + \left(\frac{\Delta\nu}{2}\right)^2} \tag{28}$$

where

$$\Delta\nu = \frac{1}{2\pi}\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) \tag{29}$$

and τ_m represents the *total* lifetime state m. The peak value of the Lorentzian lineshape function is:

$$g(\nu_0) = \frac{2}{\pi\Delta\nu} \tag{30}$$

Lifetime Broadening: Total State Lifetime

In **radiative** processes, atoms emit photons as they transition to a lower energy level. The Einstein A coefficient is related to the lifetime as

$$A_{21} = \frac{1}{\tau_{21}} \tag{31}$$

In **non-radiative** processes, atoms transition to lower energy levels without emitting a photon. These processes can be described with a rate constant k

In general, atoms may couple radiatively and non-radiatively to many lower states.

Lifetime Broadening: Total State Lifetime

A differential equation

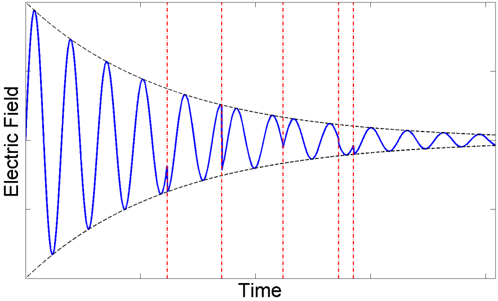
$$\frac{dN_m}{dt} = -\sum_q A_{mq} N_m - \sum_q k_{mq} N_m \tag{32}$$

By solving the above equation, the total lifetime of state m may be found to be:

$$\frac{1}{\tau_m} = \sum_q A_{mq} + \sum_q k_{mq} \tag{33}$$

Lifetime broadening is not typically the dominant broadening mechanism in gases.

Collisional Broadening: Phenomenological View



- Collisions between atoms perturb energy levels
- Effectively randomize phase
- Phase jumps broaden spectrum
- Relative times of processes in figure above not to scale

Collisional Broadening

It can be shown that in a Maxwellian gas, the frequency of collisions between two particles is:

$$\nu_{col} = N_m \sigma_{coll} \left[\frac{8 k_B T}{\pi} \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \right]^{1/2} \quad (34)$$

A more empirical description of the collisional linewidth is given by:

$$\Delta\nu = A + BP \quad (35)$$

where A is the natural linewidth B is a constant which depends on the colliding atoms. Typical values of B may be 10 MHz/torr.

Total Homogeneous Broadening

The total linewidth from all homogeneous broadening mechanisms is:

$$\Delta\nu_{total} = \frac{1}{2\pi}\left[(A_1 + k_1) + (A_2 + k_2) + 2v_{col}\right] \qquad (36)$$

where the A_m's and k_m's are the **total** radiative and non-radiative relaxation rates for level m.

Doppler Broadening I

Doppler broadening occurs in room temperature gases. In their rest frame all atoms will emit light with the same homogeneous profile, $g_H(\nu)$. However, light is **observed** in the lab frame, where the profile will be Doppler shifted.

The Doppler shift of a photon emitted from a non-relativistic particle moving toward the observer with velocity v_z is

$$\nu' = \nu\left(1 + \frac{v_z}{c}\right) \tag{37}$$

The observed lineshape profile in the lab frame is then

$$g'_H(\nu, v_z) = g_H\left(\nu\left(1 + \frac{v_z}{c}\right)\right) \tag{38}$$

Doppler Broadening II

It can be shown from thermodynamics that the distribution of particle velocities in the z direction is

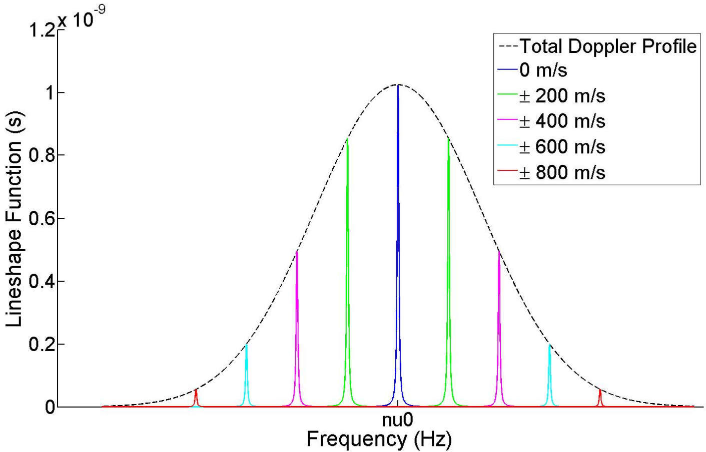
$$\left(\frac{M}{2\pi k_B T}\right)^{1/2} exp\left(-\frac{M v_z^2}{2 k_B T}\right) \tag{39}$$

The total lineshape profile in the lab frame from all particles moving with all velocities is

$$g(\nu) = \left(\frac{M}{2\pi k_B T}\right)^{1/2} \int_{-\infty}^{\infty} g_H'(\nu, v_z) \cdot exp\left(-\frac{M v_z^2}{2 k_B T}\right) dv_z \tag{40}$$

For gaseous molecules at room temperature, Doppler broadening dominates lifetime broadening, we may therefore make the approximation $g_H(\nu) \approx \delta(\nu - \nu_0)$.

Doppler Broadening Picture



Doppler Broadening III

The lineshape function of a Doppler broadened transition is:

$$g(\nu) = \frac{2}{\Delta\nu_D}\left(\frac{ln2}{\pi}\right)^{1/2} exp\left[-4\ln 2\left(\frac{\nu-\nu_0}{\Delta\nu}\right)^2\right] \tag{41}$$

where

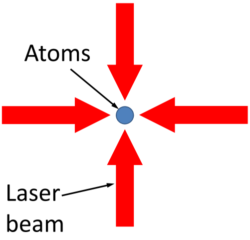
$$\Delta\nu_D = \left(\frac{8k_B T\ln 2}{Mc^2}\right)^{1/2}\nu_0 \tag{42}$$

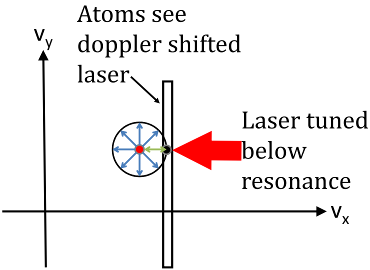
The peak of the lineshape function is:

$$g(\nu_0) = \frac{2}{\Delta\nu_D}\left(\frac{\ln 2}{\pi}\right)^{1/2} \tag{43}$$

Doppler Broadening and Laser Cooling

- Tune a laser beam to slightly below an atomic resonance
- Atoms moving toward the laser see the light Doppler shifted into resonance
- The atom is free to re-emit the photon in any direction
- Six counter-propagating beams must be used to ensure zero net momentum transfer





Example: Broadening in the Copper Vapor Laser

Problem: The copper vapor laser (CVL) operates at T = 1750 K. Its wavelength is $\lambda = 510.6$ nm and the spontaneous emission lifetime is $\tau_{sp} = 5 \times 10^{-7}$ s. Determine the form of the lineshape function and its FWHM.

Solution: The homogeneous linewidth is

$$\Delta\nu_H = \frac{1}{2\pi\tau_{sp}} = 3.18 \times 10^5 \text{ Hz} \tag{44}$$

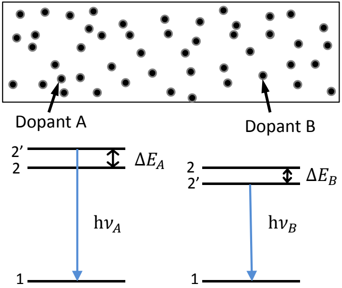
The Doppler linewidth is

$$\Delta\nu_D = \nu_0\sqrt{\frac{8k_B T \ln 2}{Mc^2}} = 2.20 \times 10^9 \text{ Hz} \tag{45}$$

Doppler broadening dominates this transition. Therefore the lineshape will be Gaussian with a FHWM $\Delta\nu_D$.

Inhomogeneous Broadening in Solid State Media

- Energy levels of transitions perturbed by local environment
- Dopants A and B encounter different perturbations due to the different concentrations and orientations of nearby dopants
- Gain profile is ensemble average of these perturbed states



Inhomogeneous Broadening: Quantum Dots

Quantum dots or "Artificial atoms"

- Energy levels determined by size and shape of dot
- Electrons and holes are confined within a small bandgap material surrounded by a material with a larger bandgap
- Light emitted when electrons and holes combine

If modeled as a 1D infinite square well, the energy of the q^{th} excited state of the electrons and holes are:

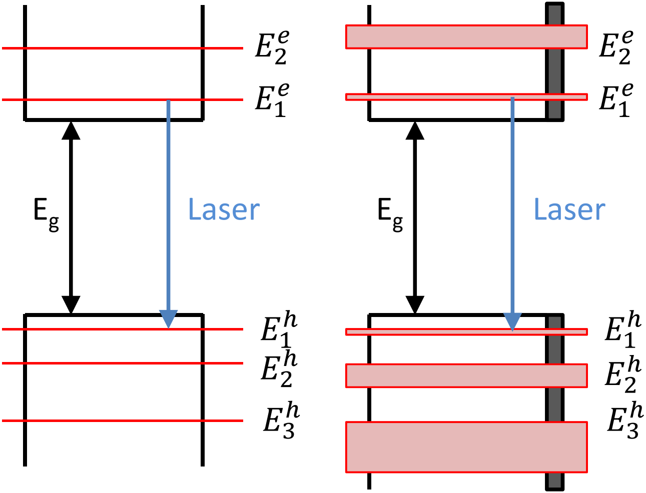
$$E_q^e = \frac{\hbar^2\pi^2 q^2}{2m_e^* L^2} \quad and \quad E_q^h = \frac{\hbar^2\pi^2 q^2}{2m_h^* L_x^2} \tag{46}$$

where m_e^* and m_h^* are the effective masses of the electrons and holes. The energy of the transition producing the photon is:

$$E = E_g + E_1^e + E_1^h \tag{47}$$

where E_g is the bandgap energy. This transition is shown on the left hand side of the next slide

Inhomogeneous Broadening: Quantum Dots



Inhomogeneous Broadening: Quantum Dots

In practice, L cannot be controlled precisely during production, so a large sample of QDs will contain dots of many sizes. Suppose lengths are distributed with a normal distribution $L \sim N(L_0, \sigma_L^2)$ or

$$p_L(L) = \left(2\pi\sigma_L^2\right)^{-1/2} \exp\left(\frac{-(L - L_0)^2}{2\sigma_L^2}\right) \tag{48}$$

Because the energy of a quantum dot depends on its size, a distribution of sizes will result in a distribution of transition energies of the quantum dots. This is shown on the right hand side of the previous slide by the grey and red boxes.

Inhomogeneous Broadening: Quantum Dots

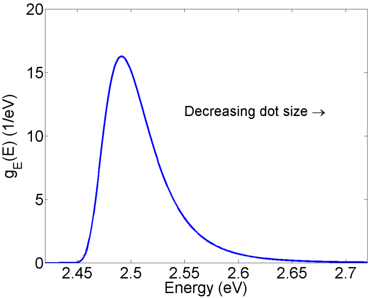
The exact lineshape may be determined by mapping the size distribution onto an energy distribution via:

$$g_E(E) = p_L(L(E)) \left| \frac{dL}{dE} \right| \quad (49)$$

where

$$L(E) = \frac{\hbar\pi}{\sqrt{2(E - E_g)}} \sqrt{\frac{1}{m_e^*} + \frac{1}{m_h^*}}$$

An example spectra is plotted on the right.

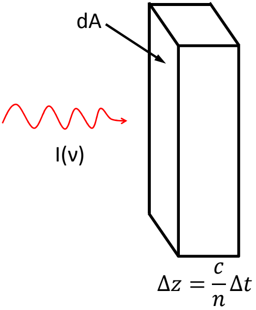


Spectra of ensemble of CdS quantum dots ($E_g = 2.42$ eV) with $L_0 = 5$ nm and $\sigma_L = 1$ nm.

Energy Density and Intensity

If all radiation is traveling in a given direction, the following relationship exists between the energy density and the intensity:

$$I(\nu) = \frac{c}{n}\rho(\nu) \qquad (50)$$



Stimulated Emission in Broadband Light

Equation 14 If the transition linewidth is much narrower than optical bandwidth, not all photons will be able to interact with the transition:

$$\begin{aligned}
\frac{dN_2}{dt} &= \int_0^\infty \left(B_{12}N_1 g(\nu)\rho(\nu) - B_{21}N_2 g(\nu)\rho(\nu)\right) d\nu && (51)\\
&= \left(B_{12}N_1 - B_{21}N_2\right)\int_0^\infty g(\nu)\rho(\nu)d\nu && (52)\\
&\approx \left(B_{12}N_1 - B_{21}N_2\right)\rho(\nu_{21})\int_0^\infty g(\nu)d\nu && (53)\\
&= \left(B_{12}N_1 - B_{21}N_2\right)\rho(\nu_{21}) && (54)
\end{aligned}$$

Here, ν_{21} is the center wavelength of the transition. In obtaining Equation 59, it has been assumed that $\rho(\nu)$ does not vary appreciably over the range in which $g(\nu)$ is non-zero.

Stimulated Emission in Narrowband Light

If the bandwidth of the optical field is much narrower than the transition linewidth, then not all atoms will be able to interact with the field.

$$\frac{dN_2}{dt} = \int_0^\infty \left(B_{12}N_1 g(\nu)\rho(\nu) - B_{21}N_2 g(\nu)\rho(\nu)\right) d\nu \quad (55)$$

$$= (B_{12}N_1 - B_{21}N_2)\int_0^\infty g(\nu)\rho(\nu)d\nu \quad (56)$$

$$\approx (B_{12}N_1 - B_{21}N_2)\, g(\nu_p)\int_0^\infty \rho(\nu)d\nu \quad (57)$$

$$= (B_{12}N_1 - B_{21}N_2)\, g(\nu_p)\rho_p \quad (58)$$

Here ν_p is the frequency of the stimulating optical field. Generally, lasers will oscillate at or very near the peak of the gain profile. It is this situation we'll be interested in for the rest of the semester.

The Possibility of Gain

Consider the interaction of a gain medium with narrowband light

$$
\begin{aligned}
\frac{dN_2}{dt} &= -B_{21}N_2\rho_\nu g(\nu) + B_{12}N_1\rho_\nu g(\nu) - \frac{N_2}{\tau_2} && (59)\\
&= -B_{21}N_2\rho_\nu g(\nu) + \frac{g_2}{g_1}B_{21}N_1\rho_\nu g(\nu) - \frac{N_2}{\tau_2} && (60)\\
&= -B_{21}\rho_\nu g(\nu)\left(N_2 - \frac{g_2}{g_1}N_1\right) - \frac{N_2}{\tau_2} && (61)\\
&= -A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu)\frac{\rho_\nu c}{n}\frac{1}{h\nu}\left(N_2 - \frac{g_2}{g_1}N_1\right) - \frac{N_2}{\tau_2} && (62)\\
&= -\frac{\sigma_{se}(\nu)I_\nu}{h\nu}\left(N_2 - \frac{g_2}{g_1}N_1\right) - \frac{N_2}{\tau_2} && (63)
\end{aligned}
$$

where:

$$\sigma_{se}(\nu) \equiv A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu) \qquad (64)$$

The Possibility of Gain II

Notice that if

$$N_2 > \frac{g_2}{g_1} N_1 \tag{65}$$

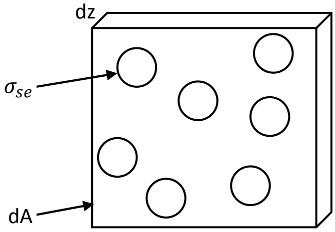
The population of the upper state will decrease. Where does the energy go? Into the optical field!

The term $N_2 - \frac{g_2}{g_1} N_1$ will appear repeatedly in our equations. To simplify equations, we define the **population inversion**

$$\Delta N \equiv N_2 - \frac{g_2}{g_1} N_1 \tag{66}$$

The Stimulated Emission Cross Section

- σ_{se} is not just a random collection of constants
- Can be viewed as 'target area' of each atom as seen by a passing photon
- If photon hits target, interaction takes place
- Higher density of atoms (targets) increases likelihood of interaction
- 'Optical thickness' $L_{opt} = 1/(\sigma_{se} N)$



Typical Values of the Stimulated Emission Cross Section

Laser	λ (nm)	σ_{se} (cm^2)
Argon Ion	488.0	2.6×10^{-12}
Nitrogen	337.1	4.0×10^{-13}
HeNe	632.8	3.0×10^{-13}
HeCd	441.6	9.0×10^{-14}
Copper Vapor	510.5	8.6×10^{-14}
GreeNe	543.5	2.0×10^{-14}
Dye (Rh6G)	577.0	2.9×10^{-16}
KrF	248.0	2.6×10^{-16}
CO_2	10600	3.0×10^{-18}
Ti:Al_2O_3	790	3.4×10^{-19}
Nd:YAG	1064	2.8×10^{-19}
Nd:Glass	1062.3	2.9×10^{-20}
Ruby	694.3	2.5×10^{-20}

Optical Amplification

Consider a plane wave propagating in the $+z$ direction through an atomic medium and recall Equation 69

$$\frac{dN_2}{dt} = -\frac{\sigma_{se}(\nu) I_\nu}{h\nu}\left(N_2 - \frac{g_2}{g_1}N_1\right) - \frac{N_2}{\tau_2} \qquad (69)$$

Neglecting non-radiative loss, every atom which leaves state 2 is converted into a photon. Therefore:

$$dN_p = \left[\frac{\sigma_{se}(\nu) I_\nu}{h\nu}\Delta N - \eta\frac{N_2}{\tau_2}\right] dt \qquad (67)$$

$$= \left[\frac{\sigma_{se}(\nu) I_\nu}{h\nu}\Delta N - \eta\frac{N_2}{\tau_2}\right] \frac{n}{c} dz \qquad (68)$$

where η is an efficiency factor. For now, assume the beam is intense enough so that the second term on the right hand side is negligible.

The Gain Coefficient

Now multiply both sides of Equation 75 by $h\nu\frac{c}{n}$ and use the fact that $I_\nu = N_p h\nu\frac{c}{n}$ to get

$$\frac{dI_\nu}{dz} = I_\nu \sigma_{se} \Delta N \tag{69}$$

This equation describes the growth/decay of an optical field in the presence of a gain medium. Equations 69 and 76 form a **nonlinear** set of differential equations.

It is customary to recognize:

$$\gamma(\nu) = \sigma_{se}(\nu)\Delta N \tag{70}$$

where γ is known as the gain. The formal definition of the gain is:

$$\gamma(\nu) \equiv \frac{1}{I_\nu}\frac{dI_\nu}{dz} \tag{71}$$

Broadening and Gain

Let us back up one step on the definition of the gain:

$$\gamma(\nu) = \sigma_{se}(\nu)\Delta N \tag{72}$$

$$= A_{21}\frac{\lambda^2(\nu)}{8\pi n^2} g(\nu)\Delta N \tag{73}$$

There are two things to take away from this equation:

- Frequency dependence of gain is primarily due to lineshape, not photon energy
- Broad transitions have lower $g(\nu)$ than narrower transitions, and hence have lower gain

Meaning of Degeneracy

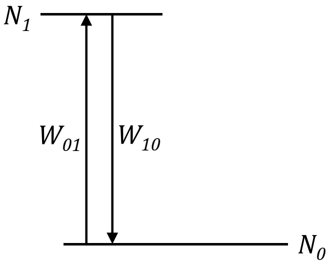
- Occurs when two energy levels have the same or nearly the same ($< k_B T$) energy
- Thermal processes will mix degenerate states
- Usually due to atomic/molecular symmetry
- Definition of 'close' can depend on equipment and circumstances

Degeneracy and Gain I

Why does degeneracy affect the gain?

- An optical field will cause both stimulated emission and absorption; It is the *net* difference between these two processes which determines whether a beam is amplified or attenuated.
- Fermi's Golden Rule (FGR) predicts a constant transition probability, W_{10}, per unit time between each individual pair of upper and lower states
- FGR also predicts that the absorption and stimulated rates are equal, so $W_{01} = W_{10}$.
- If there are multiple states at a given energy, the population at that energy will be evenly distributed among the states.

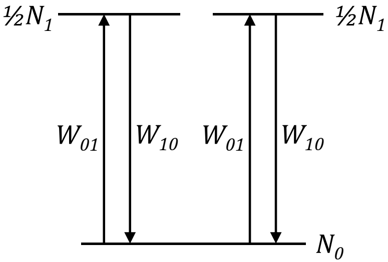
Degeneracy and Gain II



The net stimulated emission is:

$$\frac{dN_1}{dt} = -N_1 W_{10} + N_0 W_{01}$$
$$= -W_{10}(N_1 - N_0) \quad (74)$$

Degeneracy and Gain III



The net stimulated emission is:

$$\begin{aligned}\frac{dN_1}{dt} &= -\frac{N_1}{2}W_{10} + N_0 W_{01} - \frac{N_1}{2}W_{10} + N_0 W_{01} \\ &= -W_{10}\left(N_1 - 2N_0\right) \qquad (75)\end{aligned}$$

Degeneracy and Gain IV

The argument on the previous slide can be extended to any number of upper and lower states. We will always find

$$\frac{dN_1}{dt} = W_{10}\left(N_1 - \frac{g_1}{g_0}N_0\right) \tag{76}$$

where g_1 and g_2 are the upper and lower state degeneracies.

We happen to know

$$W_{10} = \frac{I\sigma_{se}}{h\nu_{10}} \tag{77}$$

Warning!! In some situations with very intense fields, the population will not have time to equilibrate among the degenerate energy levels. This analysis will then fail.

Writing Rate Equations

Rate equations keep track of state populations. We keep track of the density of atoms in a give state, so N_i has the units of cm^{-3}.

The sum of the the populations of all states is equal to the density of active atoms in the gain medium.

$$\sum_q N_q = N_{medium} \tag{78}$$

By taking the derivative of the above equation, we see

$$\sum_q \frac{dN_q}{dt} = 0 \tag{79}$$

which can be used to check if a series of rate equations is correct

Pumping Model

There are two pump models we'll use

- Constant pumping per unit volume
  - R cm^{-3}s^{-1}
  - Not typically realistic, but can be used to illustrate some properties of lasers
- Optical pumping
  - For optical pumping, from state j to state i, use

$$R = -\frac{\sigma_{se} I}{h\nu}\left[N_i - \frac{g_i}{g_j}N_j\right] \tag{80}$$

  - The negative sign is because this is written in terms of the stimulated emission rate
  - The pumping can also be written in terms of the stimulated absorption rate:

$$R = \frac{\sigma_{abs} I}{h\nu}\left[N_j - \frac{g_j}{g_i}N_i\right] \tag{81}$$

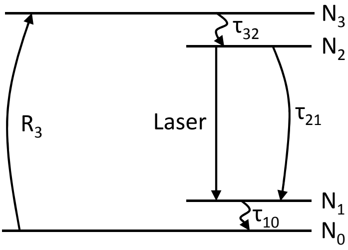
where $\sigma_{abs} = \frac{g_i}{g_j}\sigma_{se}$

n-Level Systems

- Lasers are classified based on the minimum number of energy levels necessary for an approximate analysis
- Atom and molecules have large numbers of energy states
- Simplify modeling by only including levels which contribute to lasing action

4-Level System

- The Lasing Cycle:
  1. Pump to level 3
  2. Thermally relax to level 2
  3. Lasing and spontaneous emission to level 1
  4. Thermal relaxation to level 0
- Ideally would like τ_{32} and τ_{10} to be small. *Why?*
- If $\Delta E_{10} \gg k_B T$, any pumping will create an inversion



4-Level System Rate Equations

The rate equations for the four level shown on the previous slide are

$$\frac{dN_3}{dt} = R_3 - \frac{N_3}{\tau_{32}} \quad (82)$$

$$\frac{dN_2}{dt} = \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}} - \frac{I\sigma_{se}}{h\nu}\left[N_2 - \frac{g_2}{g_1}N_1\right] \quad (83)$$

$$\frac{dN_1}{dt} = \frac{N_2}{\tau_{21}} - \frac{N_1}{\tau_{10}} + \frac{I\sigma_{se}}{h\nu}\left[N_2 - \frac{g_2}{g_1}N_1\right] \quad (84)$$

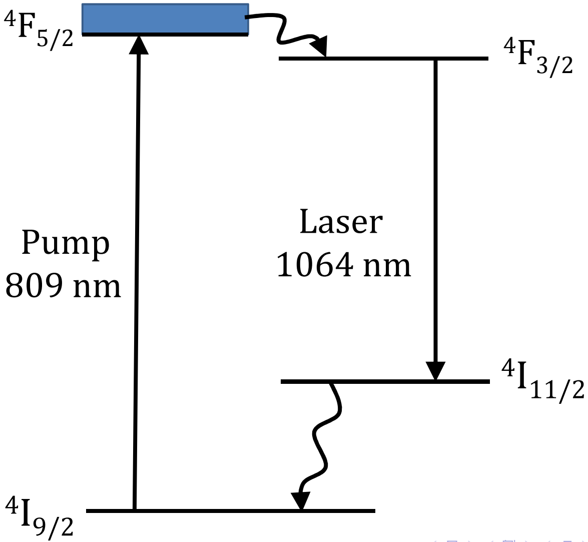
$$N_0 + N_1 + N_2 + N_3 = N_{medium} \quad (85)$$

Really, all we are interested in is the population inverison

$$\Delta N_0 = R_3\tau_{21} - \frac{g_2}{g_1}R_3\tau_{10} \quad (86)$$

What requirements must τ_{20} and τ_{10} obey to create CW laser?

Four Level System: Nd:YAG



4-Level System Rate Equations General Case

The diagram shown on slide is simplified. More generally, all levels may be pumped and there may be multiple relaxation pathways from every level. In this case, Equations 91-93 become:

$$\frac{dN_3}{dt} = R_3 - \frac{N_3}{\tau_{32}} - \frac{N_3}{\tau_{31}} - \frac{N_3}{\tau_{30}} \tag{87}$$

$$\frac{dN_2}{dt} = \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}} - \frac{N_2}{\tau_{20}} - \frac{I\sigma_{se}}{h\nu}\left[N_2 - \frac{g_2}{g_1}N_1\right] + R_2 \tag{88}$$

$$\frac{dN_1}{dt} = \frac{N_3}{\tau_{31}} + \frac{N_2}{\tau_{21}} - \frac{N_1}{\tau_1} + \frac{I\sigma_{se}}{h\nu}\left[N_2 - \frac{g_2}{g_1}N_1\right] + R_1 \tag{89}$$

This would be an ugly state diagram.

4-Level System Rate Equations General Case

The population inversion is then:

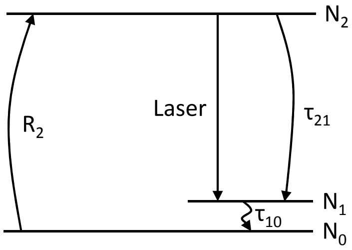
$$\Delta N_0 = \left(\frac{R_3\tau_3}{\tau_{32}} + R_2\right)\tau_2 - \frac{g_2}{g_1}\left[\left(\frac{R_3\tau_3}{\tau_{32}} + R_2\right)\frac{\tau_2}{\tau_{21}} + \frac{R_3\tau_3}{\tau_{31}} + R_1\right]\tau_1 \tag{90}$$

where $\tau_3^{-1} = \tau_{32}^{-1} + \tau_{31}^{-1} + \tau_{30}^{-1}$ and so forth. In practice, only τ_m, and not τ_{mp} are readily measured (*why?*). Try to derive Equation 99 result yourself.

In the most general case, population may escape to levels other than the 4 considered here. Stimulated emission may also be important between more than one pair of levels.

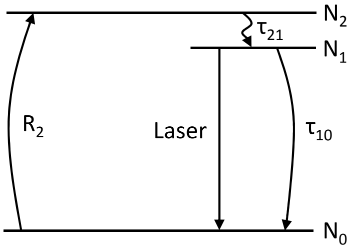
A 3-Level System I

- The Lasing Cycle:
  1. Pump to level 2
  2. Lasing and spontaneous emission to level 1
  3. Thermal relaxation to level 0
- Similar to a 4-level laser
- Ideally would like τ_{10} to be small
- If $\Delta E_{10} \gg k_B T$, any pumping will create an inversion



A 3-Level System II

- The Lasing Cycle:
  1. Pump to level 2
  2. Thermally relax to level 1
  3. Lasing and spontaneous emission to level 1
- System must be pumped to transparency before it may be pumped to inversion
- Ideally would like τ_{21} to be small



A 3-Level System II Pumping to transparency

At room temperature, the entire population sits in energy level 0. The gain on the $1 \rightarrow 0$ transition is

$$\gamma_0 = \sigma_{se} \left[N_1 - \frac{g_1}{g_0} N_0 \right] = -\sigma_{se} \frac{g_1}{g_0} N_{medium} \tag{91}$$

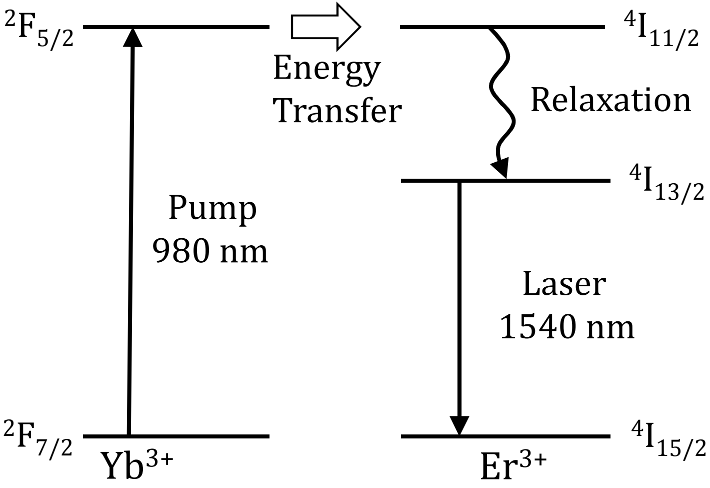
which is negative!

The power absorbed per unit volume necessary just to create $\Delta N = 0$ is:

$$P_{tr} = \frac{N_{1tr}}{\tau_1} h\nu_{20} \tag{92}$$

where $N_{1tr} = \frac{N_{medium}}{1+g_0/g_1}$. The energy used is pure waste.

Three Level System: Er Doped Fiber Amplifier



Summary

- Relationships between the Einstein A and B coefficients were established through the study of blackbody radiation
- Stimulated emission is often negligible when atoms are illuminated by a broadband light source
- Spectral broadening reduces the number of atoms able to interact with the optical field
- There are three primary mechanisms of spectral broadening in gas: lifetime, Doppler, and collisional
- Broadening mechanisms in solids and liquids are more difficult to predict
- A system will absorb light if $\Delta N < 0$ and amplify it if $\Delta N > 0$
- Rate equations model the population of various energy states in lasers.