Optical Resonator Modes

ECE 455 Optical Electronics

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ECE Illinois

Introduction

In this section, we will learn how to do the following things:

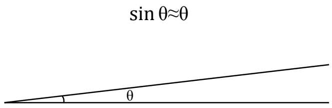
- Follow the path of paraxial light rays as they propagate through an optical system
- Determine the stability of optical resonators
- Use Gaussian beams to describe the field inside an optical resonator
- Describe the optical confinement properties of cavities with the quality factor, the finesse, and the free spectral range

ABCD Matrix I

The ABCD matrix is a ray optics formalism that relates the distance r from the optical axis and its slope r' of a ray as is propagates through optical elements.

Assumptions

1. All optical elements are "thin"
2. the angle of propagation is sufficiently small that $\sin(\theta) \approx \theta$



ABCD Matrix II

Position above optic axis and slope are represented by a vector

$$\begin{bmatrix} r \\ r' \end{bmatrix} \tag{1}$$

Cumulative effect of optical elements are matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2}$$

To find the position and slope of a ray after propagating through an optical system use

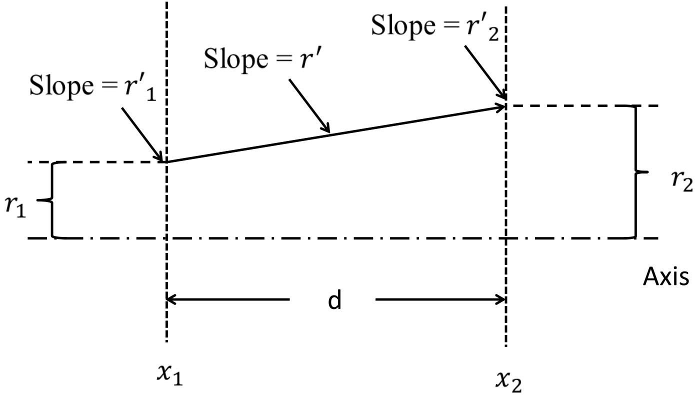
$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ r'_1 \end{bmatrix} \tag{3}$$

ABCD Matrix III

Beware!

Different books (notably Hecht and Siegman) use different conventions for ABCD matrices than defined in the previous page. Some switch the position of r and r'. Another possible convention is to multiply the slope by the refractive index.

Uniform Dielectric distance d Diagram



Uniform Dielectric distance d

It should be clear that

- The distance from the optic axis changes as:

$$r_2 = r_1 + r_1' \cdot d \tag{4}$$

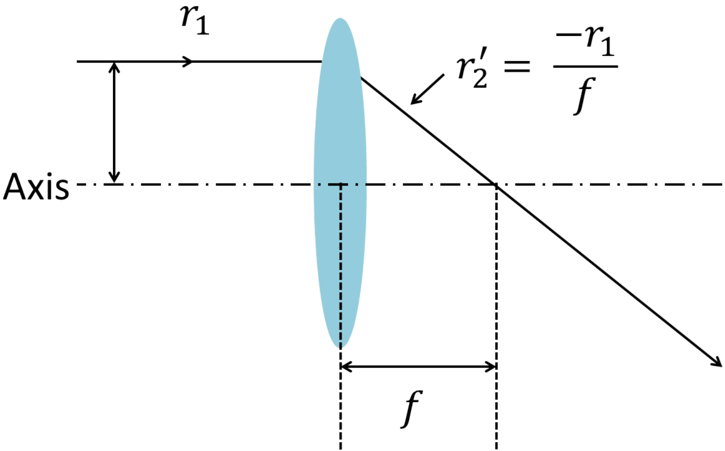
- The slope can be expected to remain constant independent of the position above or below the optic axis

$$r_2' = r_1' \tag{5}$$

- Therefore the ABCD matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \tag{6}$$

Thin Lens Diagram



Thin Lens

The ABCD matrix of the thin lens can be derived as follows:

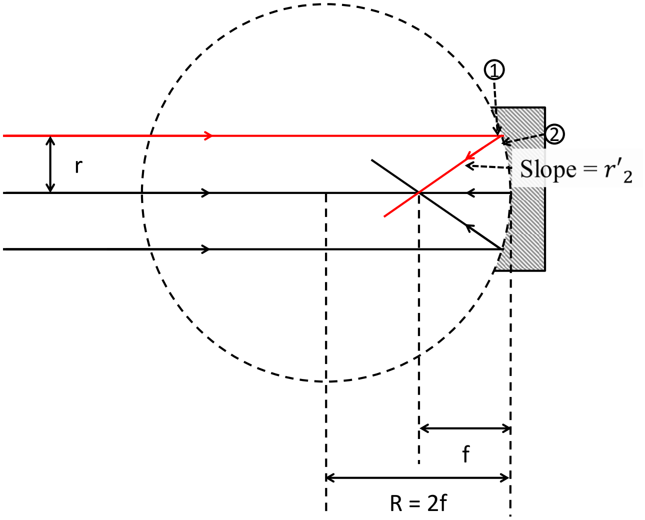
- Because the lens is thin, the distance from the axis has no chance to change while light propagates through the lens

$$r_2 \approx r_1 \tag{7}$$

- A ray traveling parallel to the optic axis ($r_1' = 0$) is focused to the origin at f. Therefore $C = -1/f$
- A ray originating at the focus of the lens $\mathbf{r} = (f \cdot r_2', r_2')^T$ will be turned parallel to the axis by the lens. Therefore: $D = 1$
- The ABCD matrix is:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \tag{8}$$

Spherical Mirror Diagram



Spherical Mirror

- Changes in the direction of propagation are ignored in ABCD matrix theory.
- Only the slope and position of the ray with respect to the axis is of interest.
- The spherical mirror is identical to the case of a thin lens. Recall that the focus of a spherical mirror with radius R is

$$f = \frac{R}{2} \tag{9}$$

- The result of the thin lens can then be used with the appropriate value substituted in place of f:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \tag{10}$$

Flat Mirror

Consider the ABCD matrix of a spherical mirror.

$$\begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \tag{11}$$

As $R \to \infty$, the mirror becomes flat. The matrix becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

The above matrix is simply the identity. The lesson: flat mirrors do not affect the path of rays.

A Several Element ABCD Matrix

To model a system consisting of several elements, the ABCD matrix for each element can be applied in series

$$\begin{bmatrix} r_{n+1} \\ r'_{n+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_n \\ r'_n \end{bmatrix} \tag{13}$$

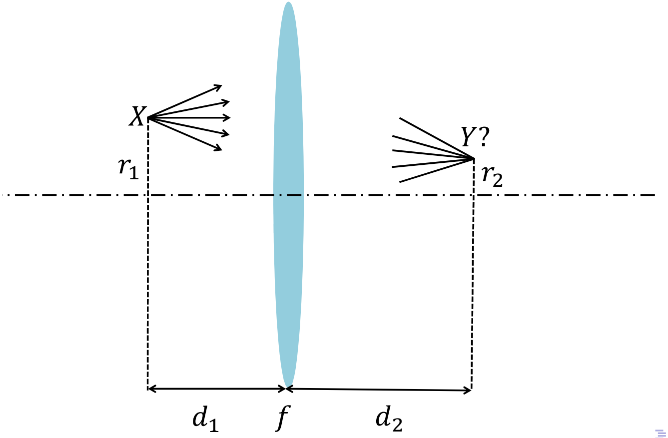
Alternatively the power of matrix multiplication can be used to find the **net** ABCD matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{bmatrix} \cdots \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \tag{14}$$

$\longleftarrow$ progression through optical system

Example: Three Elements

Problem: Light source X is placed a distance d_1 away from a lens with focal length f, as shown below. Find the point Y, where the light source is imaged.



A Three Element Example *Continued*

Solution: Point A is imaged to point B when all rays originating from point A, regardless of r_0' travel to point B.

Begin by finding the ABCD matrix. Rays travel distance d_1 along the optical axis to the lens, are refracted by the lens, and finally travel an unknown distance d_2.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{d_2}{f} & d_2 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{d_2}{f} & d_1 + d_2 - \frac{d_1 d_2}{f} \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{bmatrix} \tag{15}$$

A Three Element Example *Continued*

If element B of an ABCD matrix is zero, the initial slope doesn't matter. Setting $B = 0$ and solving for d_2 yields:

$$d_2 = \frac{1}{\frac{1}{f} - \frac{1}{d_1}} \tag{16}$$

This is of course the thin lens imaging condition.

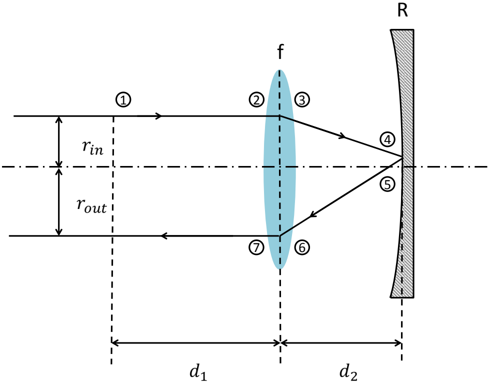
Next find the distance from the axis of the image point:

$$\begin{aligned} r_2 &= Ar_1 + Br_1' \\ &= \left(1 - \frac{d_2}{f}\right) r_1 \\ &= -\frac{d_2}{d_1} r_1 \end{aligned} \tag{17}$$

where $\frac{1}{f} = \frac{1}{d_1} + \frac{1}{d_2}$, the imaging condition, has been used.

Example: A Several Element ABCD Matrix

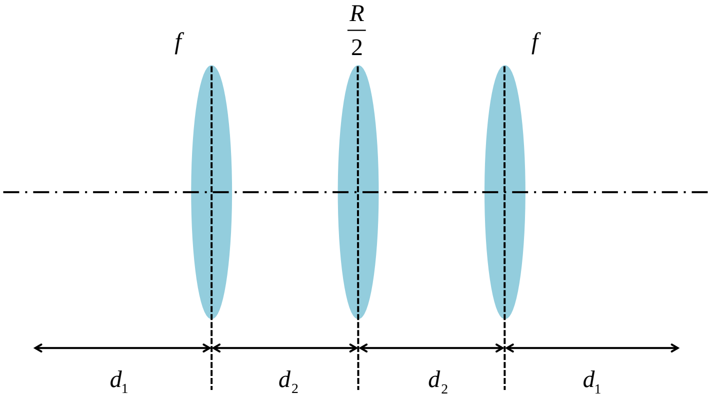
Problem: Find the output ray of the system shown below when the input ray is characterized by $r = 0.1$ cm and $r' = 0.1$



Where $f = 2$ m, $R = 4$ m, $d_1 =$5 cm and $d_2 = 5$ cm

Example: A Several Element ABCD Matrix *Continued*

Solution: The first step in solving this problem is to 'unwrap' the optical system. That leads to the following optical system below



Example: A Several Element ABCD Matrix *Continued*

The output ray can then be found with ABCD matrices. The focal lengths have been converted to centimeters.

$$\begin{aligned}
\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} &= \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \\
&\times \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} \qquad (18) \\
&= \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.005 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.005 & 1 \end{bmatrix} \\
&\times \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.005 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} \qquad (19) \\
&= \begin{bmatrix} 0.965 & 7 \\ -0.005 & 1 \end{bmatrix} \begin{bmatrix} 0.975 & 5 \\ -0.005 & 1 \end{bmatrix} \\
&\times \begin{bmatrix} 0.975 & 5 \\ -0.005 & 1 \end{bmatrix} \begin{bmatrix} -0.6 \\ -0.1 \end{bmatrix} \qquad (20)
\end{aligned}$$

Example: A Several Element ABCD Matrix *Continued*

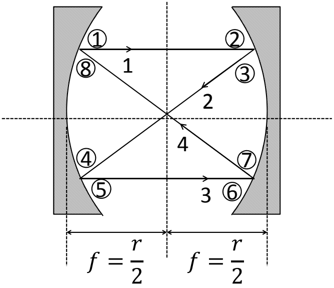
$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 0.9059 & 11.825 \\ -0.0099 & 0.975 \end{bmatrix} \begin{bmatrix} -1.085 \\ -.097 \end{bmatrix} \quad (21)$$

$$= \begin{bmatrix} -2.1299 \\ -0.0839 \end{bmatrix} \quad (22)$$

The matrix multiplication is shown in full detail above to illustrate how to more rapidly multiply matrices. Once values have been placed in the matrices with consistent units, every pair of matrices is multiplied until there is only one left.

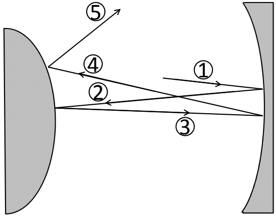
You need not only multiply the last two matrices every step!

Stable Cavity



- A **stable** optical cavity consists of two or more optical elements (usually mirrors) in which a ray will eventually replicate itself
- Have low diffraction loss
- Stable cavities have smaller mode volume
- Useful for low-gain, low-volume lasers

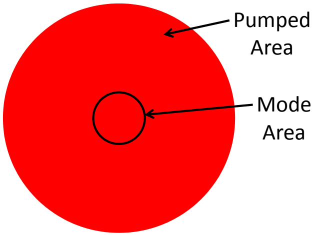
Unstable Cavity

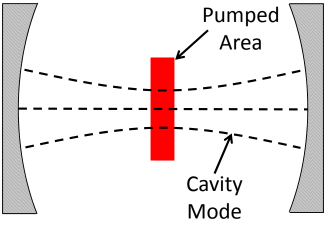


- In an **unstable** cavity, rays do not replicate themselves
- Each trip through the cavity will take the ray further from the optic axis, resulting in high diffraction losses
- Useful for high-gain, high-volume lasers

Mode-Gain Overlap

- Choice of cavity geometry affects the mode size
- Power is not extracted from areas outside the mode
- Power may go to waste
- Larger (possibly undesired) parasitic modes may begin to lase
- The lesson: the cavity should be designed so the mode overlaps the pumped region of the gain medium





ABCD Matrices and Stability

The stability of a cavity may be cast as an eigenvalue problem.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\text{round trip}} \begin{bmatrix} r \\ r' \end{bmatrix} = \lambda \begin{bmatrix} r \\ r' \end{bmatrix} \quad (23)$$

Here λ is not the wavelength, but the eigenvalue of the ABCD matrix. The eigenvalue equation is

$$(A-\lambda)(D-\lambda) - BC = 0 \quad (24)$$

This equation may be simplified by noticing for a round trip in any cavity

$$AD - BC = 1 \quad (25)$$

Why? Recall that if X and Y are matrices, then $Det(XY) = Det(X)Det(Y)$. Consider the determinants of all of the $ABCD$ matrices described in these notes.

ABCD Matrices and Stability

Placing Equation 25 into Equation 24 yields

$$\lambda^2 - (A + D)\lambda + 1 = 0 \tag{26}$$

The solution is

$$\lambda = m \pm \sqrt{m^2 - 1} \tag{27}$$

where for convenience $m \equiv \frac{A+D}{2}$. Consider the case where $|m| \leq 1$. Then m may be written as $m = \cos(\theta)$, which means

$$\lambda_\pm = \cos(\theta) \pm \imath \sin(\theta) = e^{\pm \imath \theta} \tag{28}$$

The position and slope of a ray after making n round trips through the cavity is then

$$\mathbf{r}_n = c_+ \mathbf{r}_+ e^{\imath n \theta} + c_- \mathbf{r}_- e^{-\imath n \theta} \tag{29}$$

where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the eigenvectors of the ABCD matrix

ABCD Matrices and Stability

Equation 25 can be rewritten

$$\mathbf{r}_n = \mathbf{r}_0 cos(n\theta) + \mathbf{s}_0 sin(n\theta) \tag{30}$$

Note how the maximum displacement is bounded by $\mathbf{r}_0 + \mathbf{s}_0$. In addition, because it is the sum of sines and cosines, the position will periodically repeat itself.

Our assumption that $|m| \leq 1$ implies that the **cavity is stable**. This can also be expressed as

$$-1 \leq \frac{A+D}{2} \leq 1 \tag{31}$$

Equation 31 can be cast into a different form by adding 1 to both sides and dividing the result by 2, which gives

$$0 \leq \frac{A+D+2}{4} \leq 1 \tag{32}$$

ABCD Matrices and Instability

It has been shown that if $|m| \leq 1$, the cavity is stable. It will now be shown that the opposite assumption, $|m| > 1$, implies the **cavity is unstable**.

For $|m| \geq 1$, the eigenvalues are

$$\lambda_{\pm} = m \pm \sqrt{m^2 - 1} \tag{33}$$

Both of these eigenvalues are real and positive. Therefore, after n passes through the cavity, the vector

$$r_n = c_+ r_+ e^{n\lambda_+} + c_- r_- e^{n\lambda_-} \tag{34}$$

Both terms will grow without bound.

Stability of a Two-Mirror Cavity

The ABCD matrix for a round trip of a cavity comprising two mirrors with radii R_1 and R_2 separated by a distance d is

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{2d}{R_2} & 2d - \frac{2d^2}{R_2} \\ \frac{4d}{R_1 R_2} - \frac{2}{R_1} - \frac{2}{R_2} & 1 + \frac{4d^2}{R_1 R_2} - \frac{4d}{R_1} - \frac{2d}{R_2} \end{bmatrix} \end{aligned} \tag{35}$$

If d is the mirror separation and the mirror's radii of curvature are R_1 and R_2, then the cavity will be stable if and only if

$$0 \leq \frac{\left(1 - \frac{2d}{R_2}\right) + \left(1 + \frac{4d^2}{R_1 R_2} - \frac{4d}{R_1} - \frac{2d}{R_2}\right) + 2}{4} \leq 1 \tag{36}$$

Stability of a Two-Mirror Cavity

When simplified, this expression becomes

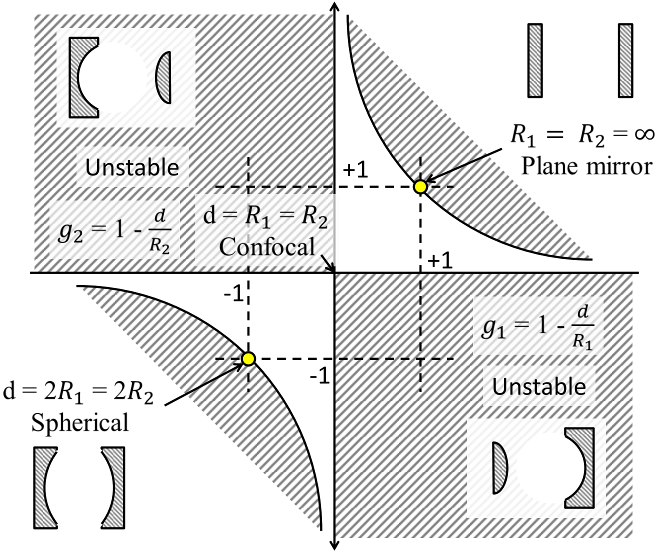
$$0 \leq \left(1 - \frac{d}{R_1}\right)\left(1 - \frac{d}{R_2}\right) \leq 1 \tag{37}$$

Often, the two terms in the product are defined as

$$g_1 \equiv 1 - \frac{d}{R_1} \quad \text{and} \quad g_2 \equiv 1 - \frac{d}{R_2} \tag{38}$$

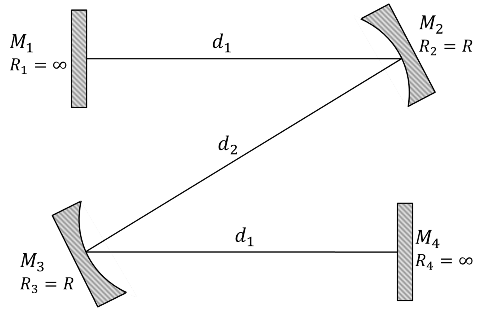
These two quantities will appear later in formulas for the eigenfrequencies of Gaussian modes. The two mirror cavity stability criterion may be plotted on a diagram, shown on the next slide.

Stability Diagram



Example: The Z-Cavity

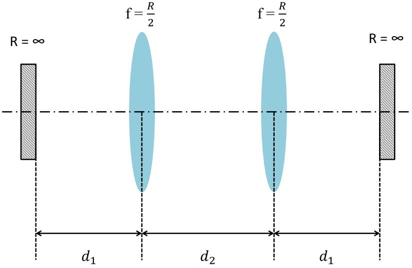
Problem: Determine the minimum radius of curvature of the two mirrors to ensure the following cavity is stable:



Solution:

The first step is to unwrap the cavity. Due to the symmetry of this cavity, it is only necessary to go from M_1 to M_4 before an equivalent position is reached.

Example: The Z-Cavity



The ABCD matrix for this half traversal of the cavity is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{1/2} = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \quad (39)$$

Example: The Z-Cavity

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{1/2} = \begin{bmatrix} 1 - \frac{4d_1+2d_2}{R} + \frac{4d_1 d_2}{R^2} & \\ & 1 - \frac{4d_1+2d_2}{R} + \frac{4d_1 d_2}{R^2} \end{bmatrix} \quad (40)$$

Here B and C have been omitted because they do not appear in the stability equation.
Using Equation 31, write

$$\left| \frac{R^2 - (4d_1 + 2d_2)R + 4d_1 d_2}{R^2} \right| \leq 1 \quad (41)$$

The final result is is that the cavity is only stable if

$$R \geq \frac{2d_1 d_2}{2d_1 + d_2} \quad (42)$$

Maxwell's Equations

Maxwell's four equations are:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (43)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad (44)$$

$$\nabla \cdot \vec{B} = 0 \quad (45)$$

$$\nabla \cdot \vec{D} = \rho \quad (46)$$

In free space or a uniform dielectric medium, $\vec{J} = 0$ and $\rho = 0$. The parameters ϵ and μ, which are material parameters, relate the fields as show below:

$$\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E} \quad (47)$$

$$\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H} \quad (48)$$

The Wave Equation

If we take the curl of Equation 43 then we may substitute Equation 44 into Equation 43 to get:

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial^2 \vec{D}}{\partial t^2} \tag{49}$$

The vector identity $\nabla \times \nabla \times \vec{E} = \nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E}$ and $\vec{D} = \epsilon \vec{E}$ can then be substituted

$$\nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\mu\epsilon \frac{\partial^2 E}{\partial t^2} \tag{50}$$

In a uniform dielectric medium, the first term on the left is zero. Then if we define $c \equiv \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ and $n \equiv \sqrt{\mu_r \epsilon_r}$, we get:

$$\nabla^2 \vec{E} = \frac{n^2}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \tag{51}$$

Plane Waves

Any field of the form:

$$\vec{E}_q(\vec{r}, t) = \vec{E}_q cos(\omega t - \vec{k}_q \cdot \vec{r} + \phi) \tag{52}$$

where $\vec{k}_q$ is the direction of propagation, $|\vec{k}_q| = \frac{\omega n}{c}$, and $\vec{E}_q \perp \vec{k}_q$ is a solution to the wave equation.

Because Maxwell's equations are linear, if $\vec{E}_q$ and $\vec{E}_{q'}$ are solutions to the wave equation, then

$$\vec{E}^t(\vec{r}, t) = \vec{E}_q(\vec{r}, t) + \vec{E}_{q'}(\vec{r}, t) \tag{53}$$

is also a solution.

Phasors

- Combining field solutions in the time domain necessitates the use of trigonometric identities.
- In monochromatic fields, the amplitude and phase of the field can be encoded in complex numbers.
- A field containing arbitrary frequency components can be created by summing monochromatic fields.
- Consider a linearly-polarized plane wave propagating in the $\hat{z}$ direction:

$$\begin{aligned} E(z,t) &= E_0 cos(-kz + \omega t + \phi) && (54) \\ &= Re\left[E_0 e^{-\imath(kz-\phi)} e^{\imath\omega t}\right] && (55) \end{aligned}$$

- The phasor of a plane wave is:

$$E(z,t) = E_0 e^{-\imath(kz-\phi)} \quad (56)$$

- Because all fields have $e^{\imath\omega t}$ time dependence, time derivatives may be replaced by $\imath\omega$

A Note on Phasor Conventions

- Many sources use $e^{\imath(kz-\omega t)}$ instead of $e^{\imath(-kz+\omega t)}$. Both are valid phasors for the same real field because cosine is an even function.
- Be careful when using equations from other sources! Many mathematical results in this section will be slightly different if the opposite convention is used instead, but the physics is the same.

Maxwell's Equations in Phasor Form

Maxwell's Equations in phasor form are then:

$$\nabla \times E = -\imath\omega B \quad (57)$$

$$\nabla \times H = \imath\omega D + J \quad (58)$$

$$\nabla \cdot \vec{B} = 0 \quad (59)$$

$$\nabla \cdot \vec{D} = \rho \quad (60)$$

The wave equation in phasor form is:

$$\nabla^2 E + \frac{\omega^2 n^2}{c^2} E = 0 \quad (61)$$

To go from phasors to the time domain, multiply the phasor by $e^{\imath\omega t}$ and take the real part.

Optical Beams

Consider an optical beam in a uniform dielectric medium propagating in the $\hat{z}$ direction. Poisson's equation is

$$\nabla \cdot \vec{E} = 0 \tag{62}$$

This can be broken up into a transverse and a longitudinal component

$$\nabla_t \vec{E}_t + \frac{\partial E_z}{\partial z} = 0 \tag{63}$$

If D is the approximate beam diameter, and $|E_t|$ is the peak field, the transverse derivative can be estimated as:

$$\nabla_t \vec{E}_t \sim \frac{|E_t|}{D} \tag{64}$$

Because the beam is primarily propagating the in $\hat{z}$ direction, the longitudinal derivative is approximately

$$\frac{\partial E_z}{\partial z} \sim -\imath \frac{2\pi n}{\lambda} E_z \tag{65}$$

Optical Beams

- The ratio of these two fields is

$$\frac{|E_z|}{|E_t|} \approx \frac{\lambda}{2\pi n D} \quad (66)$$

- For any beam with finite D, the field must have a longitudinal component.
- For a typical HeNe laser, $\lambda = 632.8$ nm and $D = 1$ mm.

$$\frac{\lambda}{2\pi D} \approx (10^{-4}) \quad (67)$$

- If $D \gg \lambda$, the longitudinal component of the field is negligible.

Paraxial Wave Equation I

- A plane wave has the same amplitude throughout all space.
- A laser beam has finite spatial extent.
- If the beam diameter is much greater than the wavelength, it propagates mostly as a plane wave and is polarized perpendicular to the direction of propagation.
- The field of a wide beam propagating in the $\hat{z}$ direction may be written as:

$$E(x, y, z) = E_0 \psi(x, y, z) e^{-\imath kz} \tag{68}$$

- $e^{-\imath kz}$ is the plane wave part of the field
- ψ represents the *small* deviation from a plane wave

Paraxial Wave Equation II

- If Eq. 68 is substituted into the phasor wave equation (Equation 61) and simplified, we obtain:

$$\nabla_t^2 \psi - 2\imath k \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{69}$$

- The last term is small compared to the other two and can therefore be neglected, leaving

$$\nabla_t^2 \psi - 2\imath k \frac{\partial \psi}{\partial z} = 0 \tag{70}$$

- At this point it is convenient to convert to cylindrical coordinates in anticipation of a rotationally symmetric solution.

Derivation of Gaussian Beam I

In cylindrical coordinates, the wave equation looks like

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \imath 2k\frac{\partial\psi}{\partial z} = 0 \quad (71)$$

An *ansatz* (guess) for the proper form of the solution which will fit the boundary conditions of a laser cavity is now introduced

$$\psi_0(r,z) = exp\left\{-\imath\left[P(z) + \frac{kr^2}{2q(z)}\right]\right\} \quad (72)$$

If this ansatz is placed in the cylindrical form the paraxial wave equation, we obtain:

$$\left[\left(\frac{k^2}{q^2(z)}\left(\frac{\partial q}{\partial z} - 1\right)\right) r^2 - 2k\left(\frac{\partial P}{\partial z} + \frac{\imath}{q(z)}\right)\right]\psi_0 = 0 \quad (73)$$

In order for the above equation to be satisfied, $\frac{\partial q}{\partial z} - 1 = 0$ and $\frac{\partial P}{\partial z} + \frac{\imath}{q(z)} = 0$

Derivation of Gaussian Beam II

- The solution to the q(z) differential equation is:

$$q(z) = z + C \tag{74}$$

- The beam has finite transverse extent, therefore ψ_0 must decay away from the optical axis.
- For this to happen, $q(z)$ must be complex. Because z is real, the constant from integration must be complex

$$q(z) = z + \imath z_0 \tag{75}$$

- At $z = 0$

$$\begin{aligned} \psi_0(r, z=0) &= exp\left(-\imath P(z=0)\right) exp\left(-\frac{kr^2}{2z_0}\right) \\ &= exp\left(-\imath P(z=0)\right) exp\left(-\left(\frac{r}{w_0}\right)^2\right) \end{aligned} \tag{76}$$

- where $w_0^2 \equiv \frac{\lambda z_0}{n\pi}$ has been defined for convenience

Derivation of Gaussian Beam III

- Expanding yields:

$$\begin{aligned}\frac{1}{q(z)} &= \frac{1}{z + \imath z_0} \\ &= \frac{z}{z^2 + z_0^2} - \frac{\imath z_0}{z^2 + z_0^2} \\ &= \frac{1}{R(z)} - \imath \frac{\lambda}{\pi n w^2(z)} \end{aligned} \tag{77}$$

where

$$R(z) \equiv z \left[1 + \left(\frac{z_0}{z}\right)^2\right] \tag{78}$$

and

$$w(z) \equiv w_0 \left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2} \tag{79}$$

have been defined because they have a physical interpretation.

Derivation of Gaussian Beam IV

Now solve the differential equation for $\frac{dP}{dz}$:

$$
\begin{aligned}
\imath P(z) &= \int_0^z \frac{dz'}{z' + \imath z_0} \\
&= ln\left[1 + \imath(z/z_0)\right] \\
&= ln\left[\left(1 + (z/z_0)^2\right)^{1/2} exp\left(-\imath tan(z/z_0)\right)\right] \\
&= ln\left[\left(1 + (z/z_0)^2\right)^{1/2}\right] - \imath tan(z/z_0) \qquad (80)
\end{aligned}
$$

The actual quantity we are interested in is:

$$
\begin{aligned}
e^{-\imath P(z)} &= \frac{1}{\left(1 + (z/z_0)^2\right)^{1/2}} e^{\imath tan^{-1}(z/z_0)} \\
&= \frac{w_0}{w(z)} e^{\imath tan^{-1}(z/z_0)} \qquad (81)
\end{aligned}
$$

Gaussian Beams I

Combining all these results[1] and placing them in our ansatz yields:

$$\begin{aligned} E(x,y,z) &= E_0 H_m\left[\frac{\sqrt{2}x}{w(z)}\right] H_p\left[\frac{\sqrt{2}y}{w(z)}\right] \frac{w_0}{w(z)} \cdot exp\left[-\frac{r^2}{w^2(z)}\right] \\ &\times exp\left[-\imath\left(kz-(1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right)\right] \\ &\times exp\left[-\imath\frac{kr^2}{2R(z)}\right] \qquad (82) \end{aligned}$$

This is a very complicated expression. We'll interpret it piece by piece.

[1] The derivation of H_m and H_n was left out for simplicity

Gaussian Beams II

First, a few definitions from the above expression

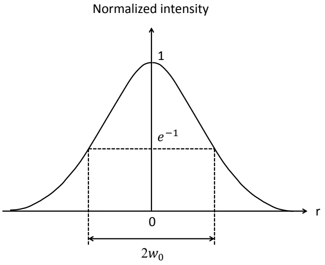
$$R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right] \tag{83}$$

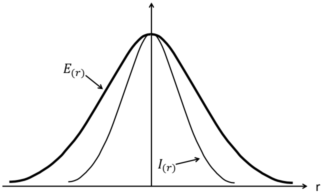
$$w(z) = w_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \tag{84}$$

$$z_0 = \frac{\pi n w_0^2}{\lambda_0} \tag{85}$$

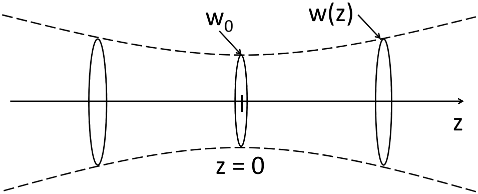
The Spot Size w(z) - Interpretation I

- Consider the following term in Equation 82 $E_0 \frac{w_0}{w(z)} exp\left[-\frac{r^2}{w^2(z)}\right]$
- $w(z)$ is known as the **spot size**
- Describes how rapidly the field decays away from the optical axis
- Characteristic radius of the beam
- The $\frac{w_0}{w(z)}$ factor is necessary for energy conservation.





The Spot Size w(z) - Interpretation II



- The beam spot size grows as $w(z) = w_0 \left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2}$
- $w(z) = w_0$ is known as the beam waist. It is the narrowest and most intense part of the beam

Spot Size w(z) - Divergence Angle I

Consider the behavior of Equation 51 as $z \to \infty$

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \tag{86}$$

$$= w_0 \frac{z}{z_0} \sqrt{1 + \left(\frac{z_0}{z}\right)^2} \tag{87}$$

$$\approx \frac{w_0}{z_0} z \left[1 + \frac{1}{2}\left(\frac{z_0}{z}\right)^2\right] \tag{88}$$

$$\approx \frac{w_0}{z_0} z \tag{89}$$

$$= \frac{\lambda_0}{\pi n w_0} z \tag{90}$$

Equation 88 was derived from Equation 87 by Taylor expansion.

Spot Size w(z) - Divergence Angle II

From the last slide we have

$$w(z) \sim \frac{\lambda_0}{\pi n w_0} z \tag{91}$$

when z is large. Recall that $m = tan(\theta)$, where m is the slope of a line and θ is the angle that the line makes with the axis. The divergence angle is defined as the total angle between the $1/e^2$ points (twice the angle made with the axis). Hence:

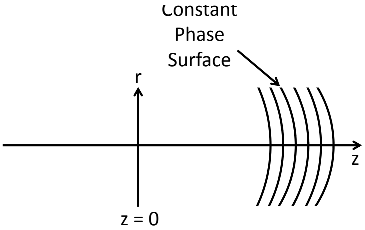
$$\theta = tan^{-1}\left(\frac{2\lambda_0}{\pi n w_0}\right) \tag{92}$$

For small slopes, $\theta \approx tan(\theta)$ and

$$\theta = \frac{2\lambda_0}{\pi n w_0} \tag{93}$$

The Radial Phase Factor R(z)

- Consider the following term in Equation 82 $exp\left[-\imath\frac{kr^2}{2R(z)}\right]$
- Surfaces of constant phase are not flat, they are parabolic
- The radius of curvature of the surfaces is $R(z)$
- Surface of constant phase at beam waist is flat $(R(0) = \infty)$
- Radius of curvature increases away from the beam waist



The Longitudinal Phase Factor

- Consider the following term in Equation 82
  $exp\left[-\imath\left(kz-(1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right)\right]$
- $e^{-\imath kz}$ is plane wave propagation in the $\hat{z}$ direction
- $e^{\imath(1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)}$ is the deviation from the plane wave velocity in the $\hat{z}$ direction because the wave must also propagate in the transverse direction
- Phase velocity of a Gaussian is lower than that of a plane wave

Transverse Modes I

- Consider the following term in Equation 82

$$H_m\left[\frac{\sqrt{2}x}{w(z)}\right] H_p\left[\frac{\sqrt{2}y}{w(z)}\right]$$

- These terms introduce structure onto the beam (see next two slides)
- Hermite-Gaussian polynomials. They can be generated with the following function

$$H_m(u) = (-1)^m e^{u^2} \frac{d^m}{du^m}\left[e^{-u^2}\right]$$

- Higher order modes occupy more volume, diverge more rapidly, and cannot be focused as tightly

m	$H_m(x)$
0	1
1	$2x$
2	$4x^2 - 2$
3	$8x^3 - 12x$

Transverse Modes II: Electric Field Profile

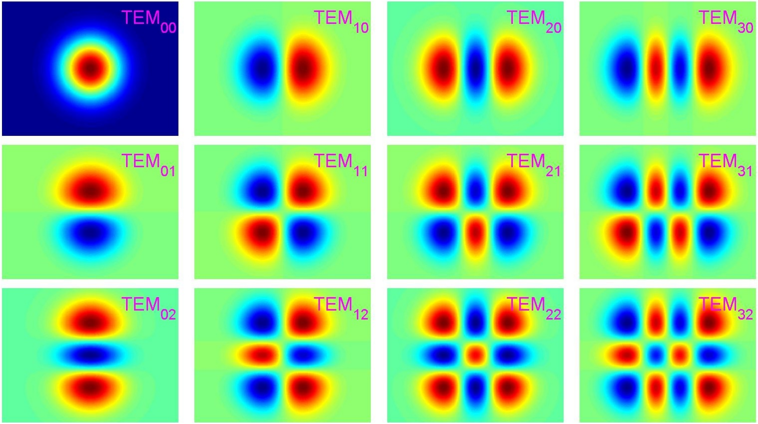


Figure: Electric field of TEM modes

Transverse Modes III: Intensity Profile

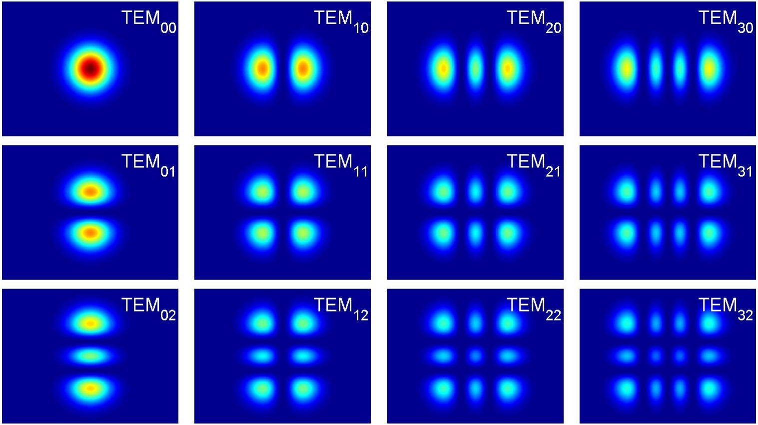


Figure: Various TEM_{mp} modes with the same w_0, normalized to have the same total optical power.

Transverse Modes IV: Laguerre-Gauss Modes

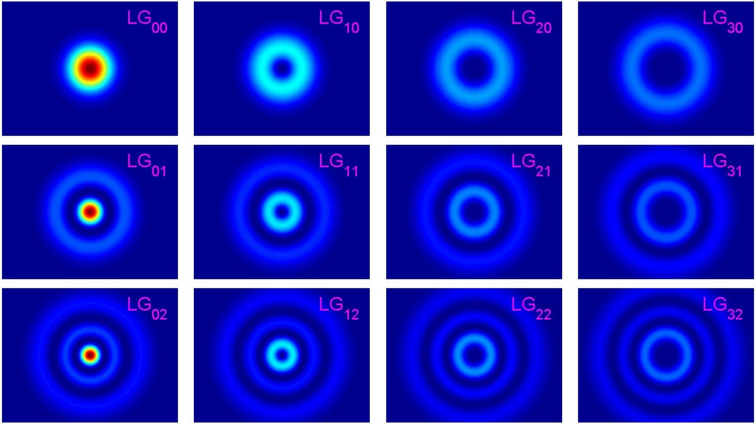


Figure: Various Gauss-Laguerre modes with the same w_0, normalized to have the same total optical power.

Laguerre-Gauss Modes

- Solutions with Hermite polynomials result from solving Equation 70 in Cartesian coordinate
- Equation 70 may also be solved in radial coordinates, resulting in Gauss-Laguerre modes
- In general, any field[2] may be written as a linear combination of Hermite-Gaussian or Gauss-Laguerre modes:

$$E_{arb} = \sum_{mp} TEM_{mp}(x, y) \tag{94}$$

$$= \sum_{\ell p} LG_{\ell p}(r, \theta) \tag{95}$$

- Infinite number of other solutions may be found in other coordinates (elliptic, hyperbolic)

[2] Provided that the transverse profile doesn't contain high enough spatial frequencies to violate the paraxial approximation

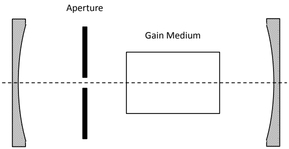
Beam Quality: M^2

- M^2 approximately measures how much faster a given beam will diffract compared to the TEM_{00} mode.
- A perfect TEM_{00} mode has $\mathrm{M}^2 = 1$.
- Gas lasers are prized for their beam quality $M^2 \sim 1$.
- M^2 is unchanged by ABCD law elements.
- Astigmatic beams have different values for M^2 along different axes: In particular for higher-order Gaussian modes

$$M_x^2 = (2m+1) \tag{96}$$
$$M_y^2 = (2p+1) \tag{97}$$

- True M^2 measurements require a fairly complex measurement device.

Forcing TEM_{00} Operation



- Multi-mode beams don't focus as tightly as the TEM_{00} mode does alone.
- Higher order modes take power away from the TEM_{00} mode.
- An aperture may be inserted which introduces loss for higher order modes.
- It is better to insert the aperture in the cavity rather than filter the output beam because the change receives positive feedback!

Forcing TEM_{00} Operation

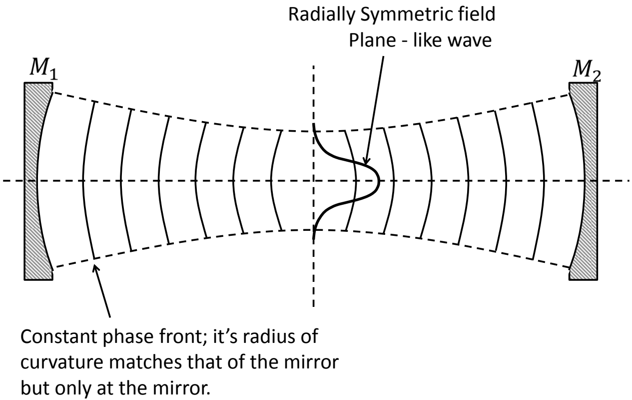
- Define cavity Fresnel number as:

$$FN \equiv \frac{a^2}{\lambda L} \tag{98}$$

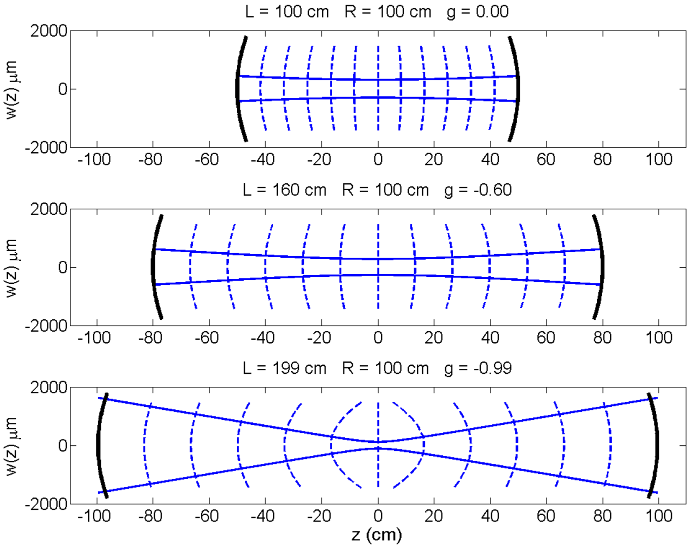
  where a is the aperture radius and L is the cavity length.
- As a rule of thumb, cavity $FN \sim 0.5$–1 is needed to force TEM_{00} operation in stable cavities.
  - $FN > 1$ allows higher order modes to oscillate with low loss
  - $FN < 0.5$ introduces high losses for fundamental mode
  - Most accurate when mirrors have high radius of curvature
  - Ultimately depends on aperture position, mirror curvature, nonlinearities, thermal gradients, gain uniformity, etc.

Gaussian Beams and Cavities

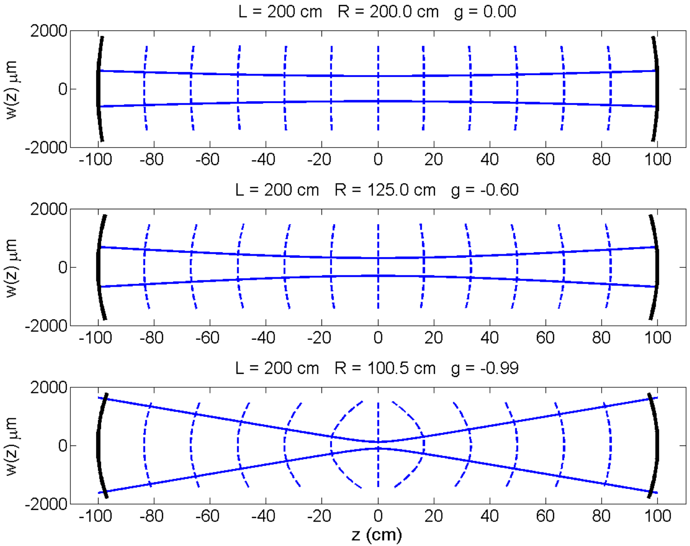
In order to be reflected unchanged, the radius of curvature of the Gaussian mode must match the curvature of the end mirrors, as illustrated below.



Gaussian Cavity Stretch



Gaussian Cavity Stretch



Gaussian Beams and Cavities

Let z_1 and z_2 represent the z-coordinates of the left and right mirror (Note this is not Verdeyen's convention), with the origin being defined as the beam waist. We can write the following three equations:

$$z_2 - z_1 = d \tag{99}$$

$$R(z_1) = z_1 \left[1 + \left(\frac{z_0}{z_1} \right)^2 \right] = -R_1 \tag{100}$$

$$R(z_2) = z_2 \left[1 + \left(\frac{z_0}{z_2} \right)^2 \right] = R_2 \tag{101}$$

Note the strange convention in Equation 100. It results from the fact that concave surfaces have a mathematically negative radius of curvature to the left of the beam waist.

Gaussian Beams and Cavities

With three equations and three unknowns, it is possible to solve the system

$$z_0^2 = \frac{d(R_1 - d)(R_2 - d)(R_1 + R_2 - d)}{(R_1 + R_2 - 2d)^2} \tag{102}$$

$$z_1 = \frac{-d(R_2 - d)}{R_1 + R_2 - 2d} \tag{103}$$

$$z_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d} \tag{104}$$

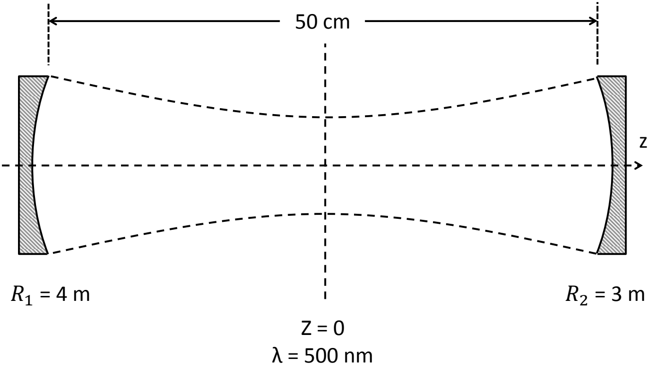
Use Your Head!

Rather than memorizing the sign convention used here (it may be different in other texts), use your common sense in applying Equation 83.

- Focusing mirrors are concave and are quoted with positive radii of curvature
- Diverging mirrors are convex and are quoted with negative radii of curvature
- The center of curvature of a Gaussian beam is always *towards* the beam waist
- The curvature is positive on the right side of the beam waist
- The curvature is negative on the left side of the beam waist

Example: Finding the Mode Inside a Cavity

Problem: Locate the beam waist in the cavity shown below. What are the spot sizes at the beam waist and at both of the mirrors?



Example: Finding the Mode Inside a Cavity

Solution: The first step is to find the spot size using Eq. 102. In keeping with the convention that focusing mirrors have a positive radius of curvature, the values to use are: $d = 50$ cm, $R_1 = +400$ cm and $R_2 = +300$ cm. The solutions for z_0 is

$$z_0 = 88.88 \text{ cm} \tag{105}$$

which can be converted to the spot size with

$$w_0 = \sqrt{\frac{\lambda z_0}{n\pi}} = 376\ \mu\text{m} \tag{106}$$

Which is the spot size at the beam waist.

Example: Finding the Mode Inside a Cavity

According to Equations 102, z_1 and z_2 are

$$z_1 = -20.83 \text{ cm} \tag{107}$$

$$z_2 = 29.17 \text{ cm} \tag{108}$$

The beam waist is therefore 20.83 cm to the right of Mirror 1. Note the beam waist is nearer the flatter mirror.

The spot sizes on the two mirrors are

$$w(z_1) = w_0 \left[1 + \left(\frac{z_1}{z_0} \right)^2 \right]^{1/2} = 386 \ \mu\text{m} \tag{109}$$

$$w(z_2) = w_0 \left[1 + \left(\frac{z_2}{z_0} \right)^2 \right]^{1/2} = 396 \ \mu\text{m} \tag{110}$$

Power Flux

Because the transverse component of the field is negligible, the intensity distribution is simply

$$I(r,z) = \frac{E^*(r,z)E(r,z)}{2\eta} \tag{111}$$

To find the flux of power out to a certain radius at a distance z away from the beam waist, use

$$P = \int_0^{r_0} I(r,z) 2\pi r \, dr \tag{112}$$

ABCD Law for Gaussian Modes

The complex beam parameter is defined as

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \imath \frac{\lambda_0}{\pi n w^2(z)} \tag{113}$$

Then the complex beam parameter at point q_2 is related to the complex beam parameter at point q_1 by:

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{114}$$

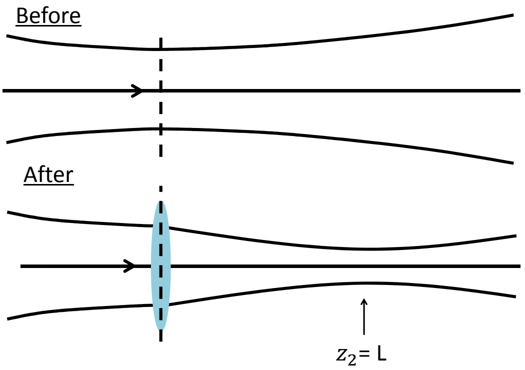
or (115)

$$\frac{1}{q_2} = \frac{C + D(1/q_1)}{A + B(1/q_1)}$$

where A, B, C and D are the coefficients of the ABCD matrix that connects the two points. See Siegman Chapter 20 for a detailed proof.

Example: Focus of a Laser Beam

Problem: A lens with focal length f is inserted at the beam waist of a Gaussian mode as shown below. Find the spot size at the new beam waist and the distance between the lens and the focus.



Example: Focus of a Laser Beam

Solution: The ABCD matrix for a lens focus f followed by a distance d is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f} & d \\ -\frac{1}{f} & 1 \end{bmatrix} \tag{116}$$

Because the radius of curvature is infinite at the beam waist, the complex beam parameter at the lens is

$$\frac{1}{q_1} = -\imath \frac{\lambda}{\pi n w_0^2} \tag{117}$$

Define $W_1 = -\frac{\lambda}{\pi n w_0^2}$ for convenience.

$$\begin{aligned} \frac{1}{q_2} &= \frac{C + \imath D W_1}{A + \imath B W_1} && (118) \\ &= \frac{C + \imath D W_1}{A + \imath B W_1} \frac{A - \imath B W_1}{A - \imath B W_1} && (119) \\ &= \frac{AC + BD W_1^2 + \imath \left(AD - BC\right) W_1}{A^2 + B^2 W_1^2} && (120) \end{aligned}$$

Example: Focus of a Laser Beam

Using the knowledge that $R = \infty$ at a beam waist, we can set the real part of the expression above to zero to find at which d the beam waist occurs.

$$AC + BDW_1^2 = 0 \tag{121}$$

After substitution, this becomes

$$\left(1 - \frac{d}{f}\right)\left(-\frac{1}{f}\right) + \left(\frac{d\lambda^2}{\pi^2 n^2 w_0^4}\right) = 0 \tag{122}$$

The final answer is

$$d = \frac{f}{1 + W_1^2 f^2} = \frac{f}{1 + \left(\frac{f\lambda}{\pi n w_0^2}\right)^2} \tag{123}$$

Which is slightly less than the focal length

Example: Focus of a Laser Beam

To find the spot size at the new focus, substitute this value of d into Equation 120

$$\frac{1}{q_2} = W_1 + \frac{1}{f^2 W_1} = \frac{\lambda}{\pi n w_0^2} + \frac{\pi n w_0^2}{\lambda f^2} \tag{124}$$

$$w_{02} = \sqrt{\frac{w_0^2 \lambda^2 f^2}{\lambda^2 f^2 + \pi^2 n^2 w_0^4}} \tag{125}$$

To make the focus as small as possible we need:

- Small λ
- Small f
- Large w_0
- Large n

Lenses with large f/d (known as $f^{\#}$) ratios are difficult (and therefore expensive) to make.

ABCD Matrices for Resonator Modes

After exactly one round trip through the cavity, the mode must repeat itself. Therefore, for one round-trip of the cavity, the beam parameter must obey:

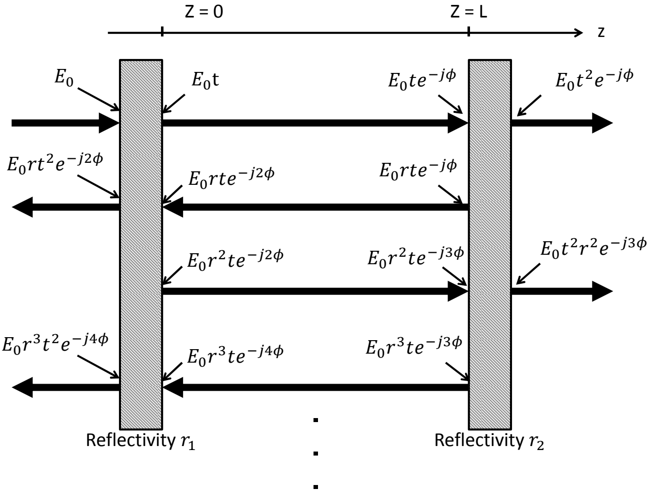
$$\frac{1}{q} = \frac{C + D(1/q)}{A + B(1/q)} \tag{126}$$

When solved for the beam parameter, we find:

$$\frac{1}{q} = \frac{-(A - D) \pm \sqrt{(A - D)^2 + 4BC}}{2B} \tag{127}$$

Note that because the round-trip ABCD matrix is different at different locations inside the cavity, the beam parameter will be different depending on where the round trip was started.

Fabry-Perot Cavity Electric Field Diagram



Fabry-Perot Electric Field

- Fabry-Perot is the simplest optical cavity – two mirrors with a uniform medium in between
- Field may be represented as infinite sum of reflected fields
- Define $\phi = nkL$
- $r_1 = r_2 = r$ has been assumed for simplicity
- r and t are field reflectivities, so conservation of energy implies $|r|^2 + |t|^2 = 1$
- $|t| = \sqrt{1 - |r|^2}$

The transmitted field is then:

$$\begin{aligned} E_t &= E_0 t^2 e^{j\phi} + E_0 t^2 r^2 e^{j3\phi} + E_0 t^2 r^4 e^{j5\phi} + \cdots \\ &= E_0 t^2 e^{j\phi} \left(1 + r^2 e^{\imath\phi} + r^4 e^{\imath 2\phi} + \cdots\right) \\ &= \frac{E_0 t^2 e^{j\phi}}{1 - r^2 e^{i2\phi}} \end{aligned} \tag{128}$$

Fabry-Perot Transmitted Intensity

The transmitted intensity is therefore:

$$I_t = \frac{1}{2\eta} E_t^* E_t \quad (129)$$

$$= \frac{I_0 T^2}{|1 - R \cdot e^{\imath 2\phi}|^2} \quad (130)$$

where $I_0 \equiv \frac{1}{2\eta} E_0^2$, $T \equiv |t|^2$, and $R \equiv |r|^2$.

Expanding the denominator yields

$$I_t = \frac{T^2}{1 - R^2} \frac{I_0}{1 + \frac{4R}{(1-R)^2} sin^2(\phi)} \quad (131)$$

Note the intensity in the cavity is larger than the transmitted field by a factor of $\frac{1}{1-R}$

Fabry-Perot Resonances

When ϕ is a multiple of π, the transmitted intensity will have a maximum. Solving $\phi = nkL = \frac{2\pi n}{\lambda_q} L = q\pi$, the wavelengths and frequencies of the q^{th} modes are determined to be:

$$\lambda_q = \frac{2nL}{q} \tag{132}$$

and

$$\nu_q = \frac{qc}{2nL} \tag{133}$$

Notice how the frequency spacing between adjacent maxima is constant. This quantity is defined as the **free spectral range** and is denoted

$$\Delta\nu \equiv \frac{c}{2nL} \tag{134}$$

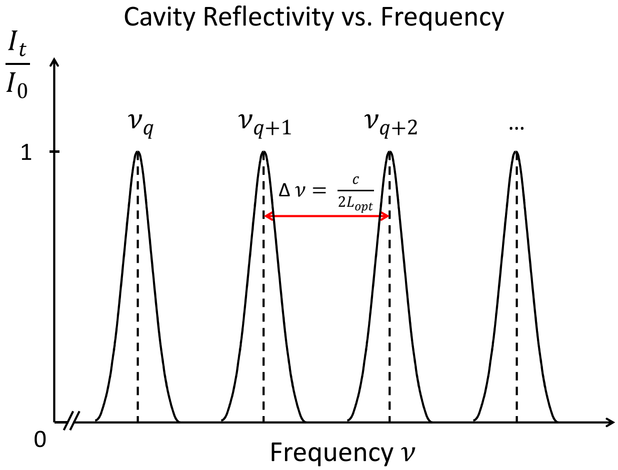
Free Spectral Range in Terms of Wavelength

Starting from $\lambda = \frac{c}{\nu}$:

$$
\begin{aligned}
d\lambda &= -\frac{c}{\nu^2} d\nu && (135) \\
&\approx -\frac{c}{\nu^2} \frac{c}{2nL} && (136) \\
&= -\frac{\lambda^2}{2nL} && (137) \\
\Delta\lambda &\approx \frac{\lambda_0^2}{2nL} && (138)
\end{aligned}
$$

This is valid when $\frac{\Delta\nu}{\nu} \ll 1$.

Fabry-Perot Transmission Profile



Example: Fabry-Perot Free Spectral Range

Problem: Find the free spectral range of a GaAs (n = 3.6, $\lambda_0 \approx 808$ nm) diode laser with end facets separated by 100 μm.
Solution:

$$\begin{aligned} \Delta\nu &= \frac{c}{2nL} \\ &= \frac{3 \times 10^8 \text{ m/s}}{2 \cdot (3.6) \cdot (1 \times 10^{-4} \text{ m})} \\ &\approx 417 \text{ GHz} \approx 0.9 \text{ nm} \end{aligned} \tag{139}$$

Problem: Find the free spectral range of an Argon Ion laser, with mirrors separated by 1.5 m and n = 1.
Solution:

$$\Delta\nu = 100 \text{ MHz} \tag{140}$$

Finesse

It can be show with a mess of algebra, that if the mirror reflectivies are unequal, the transmission through the cavity is

$$I_+ = \frac{(1-R_1)(1-R_2)}{\left(1-\sqrt{R_1R_2}\right)^2} \frac{I_0}{1+\frac{4\sqrt{R_1R_2}}{\left(1-\sqrt{R_1R_2}\right)^2} sin^2(\phi)} \quad (141)$$

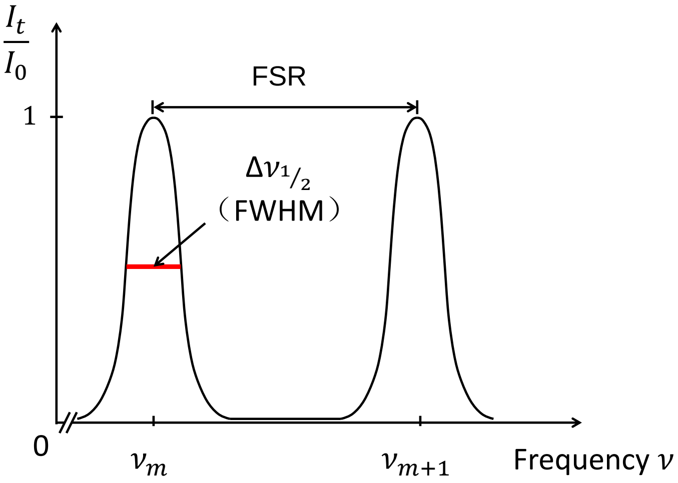
A quantity called the **finesse** is defined as the free spectral range divided by the full width at half maximum of the peak.

$$F \equiv \frac{\text{FSR}}{\text{FWHM}} \quad (142)$$

$$= \frac{c/(2nL)}{\Delta\nu_{1/2}} \quad (143)$$

$$= \frac{\pi(R_1R_2)^{1/4}}{1-(R_1R_2)^{1/2}} \quad (144)$$

A Graphical View of Finesse



The Quality Factor

Another quantity used to describe cavities is the center frequency divided by the full width at half maximum of the transmission peak.

$$Q \equiv \frac{\text{Center Frequency}}{FWHM} \tag{145}$$

$$= \frac{\nu_0}{\Delta\nu_{1/2}} \tag{146}$$

$$= \frac{\lambda_0}{\Delta\lambda_{1/2}} \tag{147}$$

$$= \frac{2\pi nL}{\lambda_0} \frac{(R_1 R_2)^{1/4}}{1-(R_1 R_2)^{1/2}} \tag{148}$$

In crystal oscillator RF circuits, Q can be $10^5 - 10^6$. In laser cavities, Q can be in excess of 10^8!!

Further Interpretations of the Quality Factor

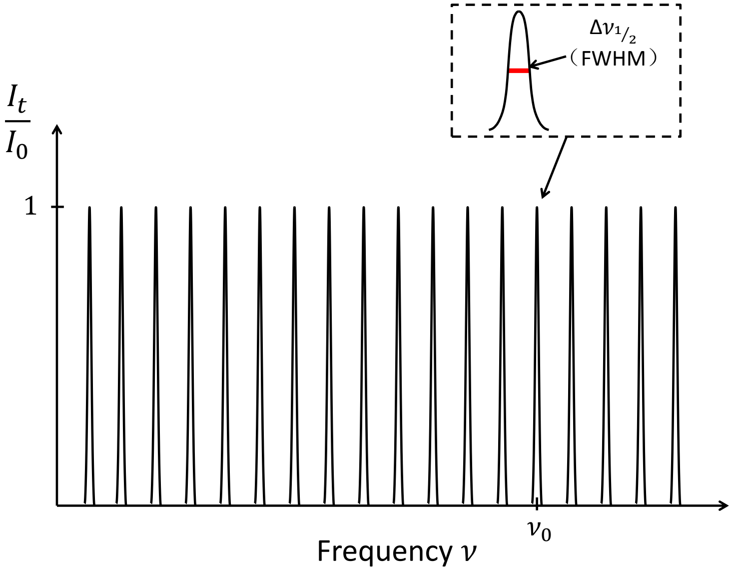
- The quality factor can be related directly to cavity parameters, rather than the cavity's spectral characteristics:

$$Q = 2\pi \frac{\text{Max Stored Energy}}{\text{Energy Lost Per Cycle}} \quad (149)$$

$$= 2\pi\nu_0 \frac{\text{Max Stored Energy}}{\text{Average Power Loss}} \quad (150)$$

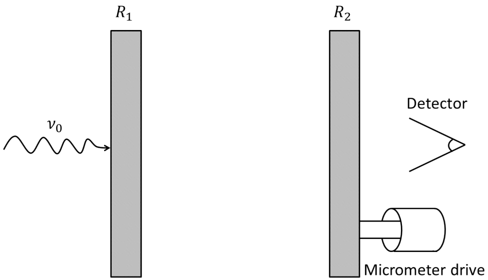
- Loss may come from scattering or absorption in the cavity or light coupling out of the cavity
- Q may also be interpreted as the number of oscillations observed before the amplitude of the oscillation decays below $e^{-2\pi}$

A Graphical View of the Quality Factor



Example: FSR, Q, F

Problem: Find the FSR, Q, and F of the cavity shown below at a wavelength of 1 μm.



Solution: The free spectral range is

$$FSR = \frac{c}{2nL} = 150 \text{ GHz} \tag{151}$$

Example: FSR, Q, F *Continued*

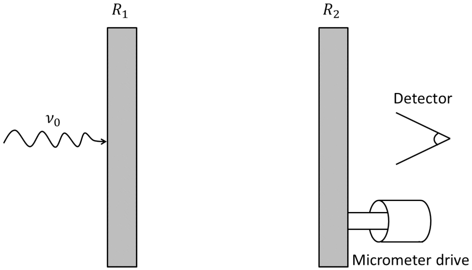
The quality factor of the cavity is

$$Q = \frac{2 \cdot (0.001 \text{ m})\pi(0.995)^{1/2}}{(1 \times 10^{-6} \text{ m})(1 - 0.995)} \approx 1.25 \times 10^6 \tag{152}$$

Its finesse is

$$F = \frac{\pi(0.995)^{1/2}}{1 - 0.995} = 627 \tag{153}$$

Fabry-Perot Tuning: Changing Mirror Separation



- Fabry-Perot cavities can be used to examine fine spectral features.
- A length change of $\lambda/4$ will shift a cavity from resonance to anti-resonance
- $\frac{\Delta L}{L} \ll 1$ for the FSR to remain constant over tuning range

Fabry-Perot Tuning: Changing Mirror Separation

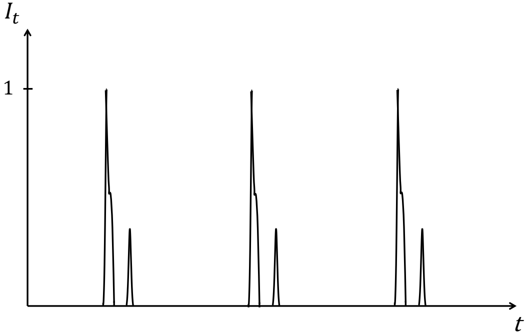
Recall that if ϕ is the phase shift from a one way pass through the cavity, the intensity of light transmitted through a FP cavity is:

$$\begin{aligned} I_t &\propto \frac{1}{1+\frac{4R}{(1-R)^2} sin(\phi)} \\ &= \frac{1}{1+\frac{4R}{(1-R)^2} sin(nkL)} \\ &= \frac{1}{1+\frac{4R}{(1-R)^2} sin(\frac{2\pi}{c}\nu nL)} \end{aligned} \tag{154}$$

The last equation tells us that a plot of transmitted intensity will have the same shape regardless of whether we sweep the index of refraction, the frequency of the incident light, or the length of the cavity while holding the other two constant. Therefore the finesse may be esimated from any of these plots.

Example: Fine Spectral Features

Problem: A CW HeNe laser is shone upon a FP cavity with a FSR of 4.25 GHz. One mirror of the FP is rapidly moved with a piezo-electric actuator. The transmitted intensity as a function of time is shown below. What is the Finesse of the Fabry-Perot cavity? How far apart are the end mirrors of the HeNe laser?



Example: Fine Spectral Features

Solution: The first step is to interpret the data:

- The periodic pattern is from the same frequencies going in and out of resonance as the FP cavity is tuned
- The distance from the one of the high peaks to the next is the free spectral range of the FP cavity
- Assume the taller and shorter peaks are adjacent longitudinal modes of the HeNe. The distance between them is then one FSR of the HeNe
- Not quite resolved next to the larger peak is a higher-order transverse mode.

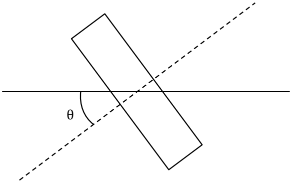
The finesse can be estimated by simply finding the ratio of the FSR to the FWHM found by measuring the figure. A good guess is $F = 45$.

Example: Fine Spectral Features

In the same way, the frequency separation between the two longitudinal modes is estimated to be 500 MHz.

$$L = \frac{c}{2 \cdot FSR} = \frac{c}{2 \cdot 500 MHz} = 30 \text{ cm} \tag{155}$$

Fabry-Perot Tuning: Tilting



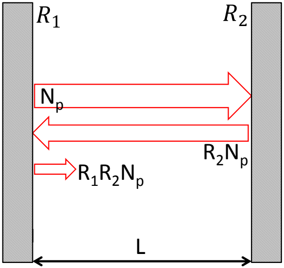
If the cavity is tilted with respect to the incident beam, the phase shift per round trip is no longer $2kL$. Instead the phase shift is:

$$\phi' = \frac{2kL}{sin(\theta)} \tag{156}$$

Fine tuning of a laser's wavelength frequently accomplished by inserting a high-Q Fabry-Perot etalon into a larger cavity.

Photon Lifetime

Consider a cavity such as the one shown below



After each trip through the cavity, only a fraction S $(0 \le S \le 1)$ of photons remain. Hence if there are N_p photons in the cavity initially, the change in the number of photons in the cavity is

$$\Delta N_p = -(1 - S) \cdot N_p \tag{157}$$

Photon Lifetime

The time required for light to make one round trip in the cavity is

$$t_{RT} = \frac{2nL}{c} \tag{158}$$

The change in the number of photons per unit time is

$$\begin{aligned} \frac{dN_p}{dt} &\approx \frac{\Delta N_p}{\Delta t} \\ &= -\frac{(1-S)N_p}{t_{RT}} \end{aligned} \tag{159}$$

The familiar solution to this equation is

$$N_p(t) = N_{p0} e^{-\frac{t}{\tau_p}} \tag{160}$$

Where τ_p is called the photon lifetime and is defined as

$$\tau_p = \frac{t_{RT}}{1-S} \tag{161}$$

Survival

Most generally, the photon lifetime is defined as

$$\tau_p = \frac{\text{round trip time}}{\text{fractional loss per round trip}} \tag{162}$$

The loss may come from several sources

- Finite mirror reflectivity
- Unsaturated intracavity absorption
- Scattering loss
- Diffraction loss

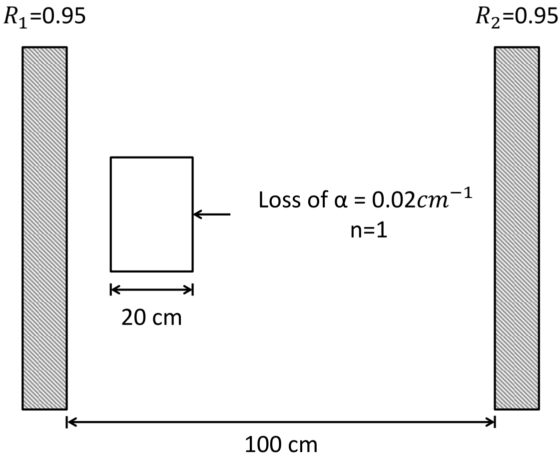
A Two-Mirror Cavity

Consider the following cavity with end mirrors separated by L and reflectivities R_1 R_2. The loss of the cavity is dominated by the finite reflectivity of the mirrors. In this case the survival rate is $S = R_1 R_2$. The round trip time is $t_{RT} = \frac{2nL}{c}$, which means the photon lifetime is

$$\tau_c = \frac{2nL}{c \cdot (1 - R_1 R_2)} \tag{163}$$

Example: Cavity with an Absorber

Problem: Determine the photon lifetime of the cavity below.



Example: Cavity with an Absorber

Solution: The first step should be to determine the survival fraction of photons during one round trip

$$S = R_1 exp(-\alpha L) R_2 exp(-\alpha L) = 0.4055 \tag{164}$$

Note that during the round trip, light must make two trips through the absorber. Next the round trip time is

$$t_{RT} = \frac{2nL}{c} = 6.67 \text{ ns} \tag{165}$$

Going back to Equation 162, the photon survival time is

$$\tau_p = \frac{6.667 \text{ ns}}{1 - 0.4055} = 11.2 \text{ ns} \tag{166}$$

A Useful Relation

Cavity Q is related to the cavity lifetime, τ_c, and angular frequency of light, ω, by:

$$Q = \omega\tau_c \tag{167}$$

This useful relation allows measurement of the cavity lifetime **without** performing a temporal measurement.

General Cavity Resonances

The condition for resonance for any mode is:

$$\int_{RT} k_z dz = q2\pi \tag{168}$$

which states that the phase shift during a round trip must be an integer multiple of 2π. It is a consequence of the fact that the field must be single-valued at every point.

Consider a plane wave in a cavity with a non-uniform index of refraction. We can use $k_z = \frac{2\pi\nu_q n(z)}{c}$ in Equation 168 to get

$$\nu_q = \frac{qc}{\int_{RT} n(z)dz} \tag{169}$$

as the resonance frequencies of the cavity.

Resonant Wavelengths of Gaussian Modes

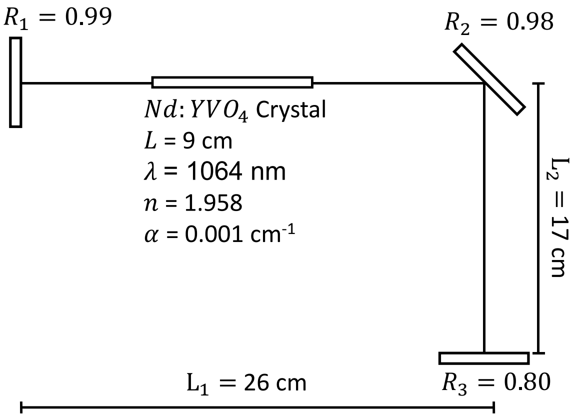
Applying Equation 168 to Equation 82, we find the eigenfrequencies the Gaussian modes within a resonator are:

$$\nu_{m,p,q} = \frac{c}{2nd}\left[q + \frac{1+m+p}{\pi} cos^{-1}\sqrt{g_1 g_2}\right] \tag{170}$$

where m and p are the transverse quantum number, q is the longitudinal quantum number, and $g_i = \left(1 - \frac{d}{R_i}\right)$ is the stability parameter.

Example: Cavity with an Absorber

Problem: Find the FSR, F, τ_p, and Q for the cavity shown below.



Example: Cavity with an Absorber

Solution: First, we must find the optical path length

$$\begin{aligned} L_{opt} &= 17 \text{ cm} + (26 - 9) \text{ cm} + 1.958 \cdot 9 \text{ cm} \\ &= 51.62 \text{ cm} \end{aligned} \tag{171}$$

Equation 169 tells that the FSR is:

$$FSR = \frac{c}{2L_{opt}} = 290.4 \text{ MHz} \tag{172}$$

To find the finesse, we can modify Equation 144

$$F = \frac{\pi(R_1 R_2^2 R_3 e^{-2\alpha L})^{1/4}}{1 - (R_1 R_2^2 R_3 e^{-2\alpha L})^{1/2}} = 21.53 \tag{173}$$

Example: Cavity with an Absorber

The photon lifetime is:

$$\tau_p = \frac{2L_{opt}}{c(1 - R_1 R_2^2 R_3 e^{-2\alpha L})} = 13.6 \text{ ns} \tag{174}$$

Finally, the cavity Q is:

$$Q = \frac{2L_{opt}}{\lambda_0} \frac{\pi (R_1 R_2^2 R_3 e^{-2\alpha L})^{1/4}}{1 - (R_1 R_2^2 R_3 e^{-2\alpha L})^{1/2}} = 2.09 \times 10^7 \tag{175}$$

Non-Traditional Cavities

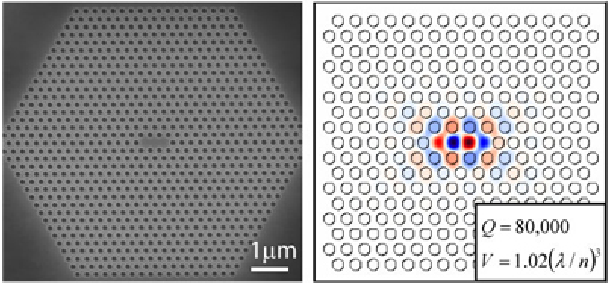


Figure:
Source: http://www.ireap.umd.edu/NanoPhotonics/PCdevices.html

- Not all optical cavities consist of mirrors
- Any structure which confines light may behave as a laser cavity

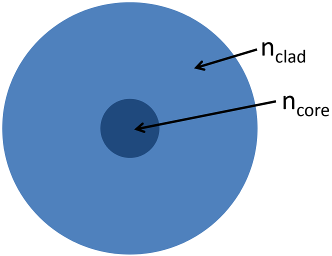
Fiber Cavity

- Light may be confined in a waveguide with $n_{core} > n_{clad}$
- The electric field for such a waveguide is:

$$E_{core}(r, \phi, z) = E_0 J(k_r r) e^{\imath m\phi} e^{-\imath k_z z}$$

where $k_r = \sqrt{n_{core}^2 k_0^2 - k_z^2}$

- Mirrors on end of fiber form a cavity
  - Reflection from index change at end of fiber can also be considered a mirror



Summary

- The paths of paraxial light rays through optical systems can be described with ABCD matrices
- Determine the stability of optical resonators
- The electric field inside an optical cavity is described with Gaussian beams
- Describe the optical containment properties of cavities with the quality factor, the finesse, and the free spectral range

Useful Matrix Identities

Let X, Y, Z, and W be matrices. Then the following properties hold

$(XY)Z = X(YZ)$	Associative Property
$det(XY) = det(X)det(Y)$	
$XYZW = (XY)(ZW)$	
$XY \neq YX$	Non-commutativity