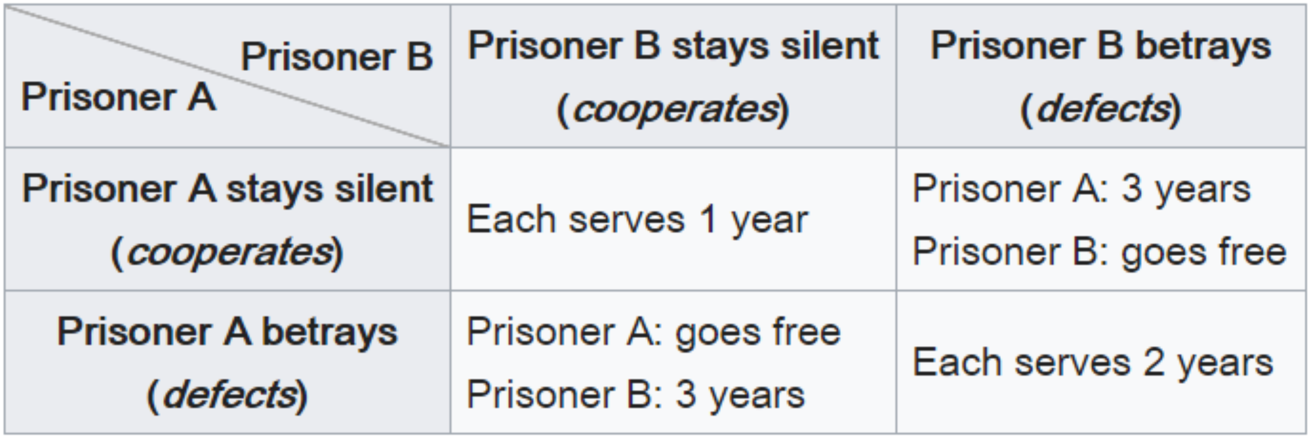
CS 440/ECE448 Lecture 23: Game Theory

Mark Hasegawa-Johnson, 3/2024



Prisoner B Prisoner A	Prisoner B stays silent (*cooperates*)	Prisoner B betrays (*defects*)
Prisoner A stays silent (*cooperates*)	Each serves 1 year	Prisoner A: 3 years Prisoner B: goes free
Prisoner A betrays (*defects*)	Prisoner A: goes free Prisoner B: 3 years	Each serves 2 years

https://en.wikipedia.org/wiki/Prisoner’s_dilemma

Today: Games with Simultaneous Moves

Assume:

- two-player game, deterministic environment (not necessary, but simplifies the problem),
- rational players (each player tries to maximize their own reward),
- not zero-sum (game can have 0, 1, or 2 winners),
- simultaneous moves.

Some surprising results:

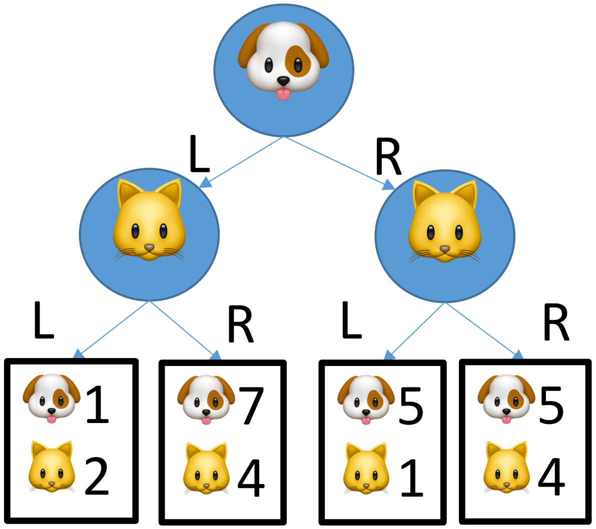
1. The rational course of action changes may depend on your belief about what the other player will do (Nash equilibrium).
2. There are different ways to define “optimum” (Pareto optimal outcomes).
3. There may be a Pareto optimal outcome that a rational player is forced to reject (Dominant strategy).
4. In some cases, the rational thing to do is to play randomly (Mixed-strategy equilibrium).

Outline of today’s lecture

- Games with simultaneous moves: Notation
- Example: Stag Hunt (Coordination Games)
  - Nash Equilibrium: Each player knows what the other will do, and responds rationally
- Example: Asymmetric Coordination Games
  - Pareto Optimal outcome: No player can win more w/o some other player winning less
- Example: Prisoners’ Dilemma (Betrayal Games)
  - Dominant Strategy: an action that is rational regardless of what the other player does
- Example: Chicken (Anti-Coordination Games)
  - Randomness can be rational: Mixed Nash Equilibrium

Notation: sequential games

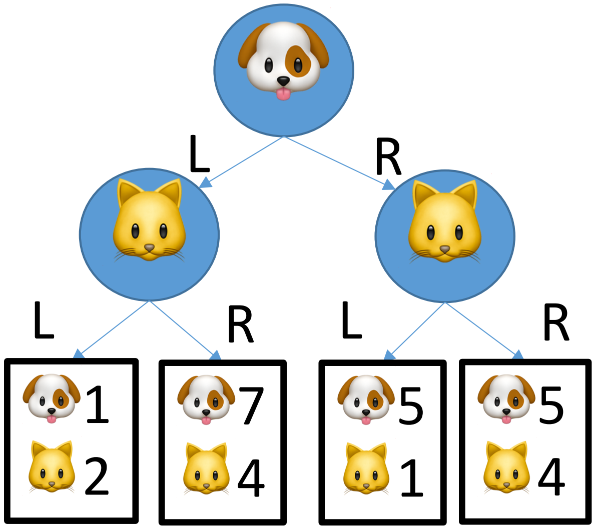
- Players take turns acting (e.g., dog moves first, then cat)
- Each node represents the action of one player (e.g., each animal can go either L or R)
- Terminal node is marked with the value for each player

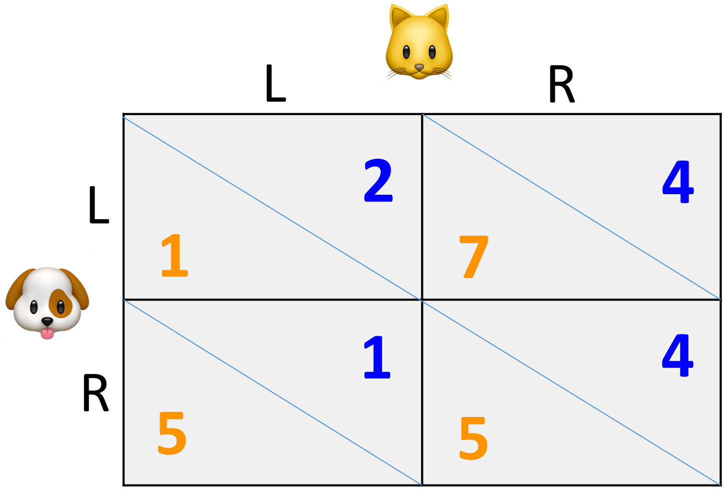


Notation: simultaneous games

The payoff matrix shows:

- Each column is a different move for player 1.
- Each row is a different move for player 2.
- Each square is labeled with the rewards earned by each player in that square.





		🐱	
		L	R
🐶	L	2 1	4 7
	R	1 5	4 5

Payoff matrix

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Stag hunt

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Defect	**Cooperate**
Bob	**Defect**	10 10	0 10
	Cooperate	10 0	100 100

By Ancheta Wis, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=68432449

Apparently first described by Jean-Jacques Rousseau:

- If both hunters (Bob and Alice) cooperate in hunting for the stag → each gets to take home half a stag (100lbs)
- If one hunts for the stag, while the other wanders off and bags a hare → the defector gets a hare (10lbs), the cooperator gets nothing.
- If both hunters defect → each gets to take home a hare.

Nash Equilibrium

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Defect	**Cooperate**
Bob	**Defect**	10 10	0 10
	Cooperate	10 0	100 100

By Ancheta Wis, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=68432449

A Nash Equilibrium is a game outcome such that each player, **knowing the other player’s move in advance**, responds rationally.

Nash Equilibrium

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Defect	**Cooperate**
Bob	**Defect**	10 10	0 10
	Cooperate	10 0	100 100

By Ancheta Wis, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=68432449

Example: (Defect,Defect) is a Nash equilibrium.

- Alice knows that Bob will defect, so she defects.
- Bob knows that Alice will defect, so he defects.
- Neither player can ***rationally*** change his or her move, unless the other player also changes.

Nash Equilibrium

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Defect	Cooperate
Bob	Defect	10 10	0 10
	Cooperate	10 0	100 100

By Ancheta Wis, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=68432449

(Cooperate,Cooperate) is also a Nash equilibrium!

- Alice knows that Bob will cooperate, so she cooperates!
- Bob knows that Alice will cooperate, so she cooperates!
- Neither player can ***rationally*** change his or her move, unless the other player also changes.

Surprising result #1: Nash equilibrium depends on belief

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Defect	Cooperate
Bob	Defect	10 10	0 10
	Cooperate	10 0	100 100

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Surprising result:

The rational course of action depends on what you believe the other player will do.

How is "belief" formed? Answer: usually, by watching them play the game against other players, and observing their usual policy.

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- Example: Chicken (Anti-Coordination Games)
  - Randomness can be rational: Mixed Nash Equilibrium

Asymmetric Coordination Games

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Stag	**Alligator**
Bob	**Stag**	10 20	0 0
	Alligator	0 0	20 10

By Tartarus is a n00b, public domain, https://commons.wikimedia.org/wiki/File:AmericanAlligator.JPG

Alice prefers alligator. Bob prefers stag.

If they don’t cooperate, they each get nothing.

Asymmetric Coordination Games

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Stag	Alligator
Bob	Stag	10 20	0 0
	Alligator	0 0	20 10

By Tartarus is a n00b, public domain, https://commons.wikimedia.org/wiki/File:AmericanAlligator.JPG

The Nash equilibria are (Stag,Stag) and (Gator,Gator).

- If Bob knows that Alice will hunt gator, then it’s rational for him to do the same.
- If Alice knows that Bob will hunt stag, then it’s rational for her to do the same.

What happens if they trust one another?

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Stag	**Alligator**
Bob	**Stag**	**10** **20**	**0** **0**
	Alligator	**0** **0**	**20** **10**

By Tartarus is a n00b, public domain, https://commons.wikimedia.org/wiki/File:AmericanAlligator.JPG

What happens if they discuss their actions, and make promises, and trust one another?

It depends: whose needs are considered more important?

- If Bob’s needs are more important, then they will hunt stag.
- If Alice’s needs are more important, then they will hunt alligator.

Pareto optimal outcome

Photo by Scott Bauer, Public Domain, https://commons.wikimedia.org/w/index.php?curid=245466

		Alice	
		Stag	Alligator
Bob	Stag	10 20	0 0
	Alligator	0 0	20 10

By Tartarus is a n00b, public domain, https://commons.wikimedia.org/wiki/File:AmericanAlligator.JPG

An outcome is ***<u>Pareto-optimal</u>*** if the only way to increase value for one player is by decreasing value for the other.

- (Stag,Stag) is Pareto-optimal: one could increase Alice's value, but only by decreasing Bob's value.
- (Alligator,Alligator) is Pareto-optimal: one could increase Bob's value, but only by decreasing Alice's value.

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Prisoner’s dilemma

- Two criminals have been arrested and the police visit them separately
- If one player testifies against the other and the other refuses, the one who testified goes free and the one who refused gets a 10-year sentence
- If both players testify against each other, they each get a 5-year sentence
- If both refuse to testify, they each get a 1-year sentence

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	Alice: Testify	**Alice: Refuse**
Bob: Testify		
Bob: Refuse		

Prisoner’s dilemma

- Two criminals have been arrested and the police visit them separately
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By Monogram Pictures, Public Domain, https://commons.wikimedia.org/w/index.php?curid=50338507

	Alice: Testify	**Alice: Refuse**
Bob: Testify	-5 -5	-10 0
Bob: Refuse	0 -10	-1 -1

Questions that can be asked

- If you were permitted to discuss options with the other player, but if one of you is more persuasive than the other, what are the different possible outcomes that might result from that discussion?
- If you knew in advance what your opponent was going to do, what would you do?
- If you didn’t know in advance what your opponent was going to do, what would you do?

Pareto optimality

If you were permitted to discuss options with the other player, what are the different possible outcomes that might result from that discussion?

- If Bob's needs are considered most important, the (-10,0) outcome might result.
- If Alice's needs are considered more important, the (0,-10) outcome might result.
- If their needs are equally important, the (-1,-1) outcome might result.

A ***Pareto optimal*** outcome is an outcome whose cost to player A can only be reduced by increasing the cost to player B.

	Alice: Testify	**Alice: Refuse**
Bob: Testify	-5 -5	-10 0
Bob: Refuse	0 -10	-1 -1

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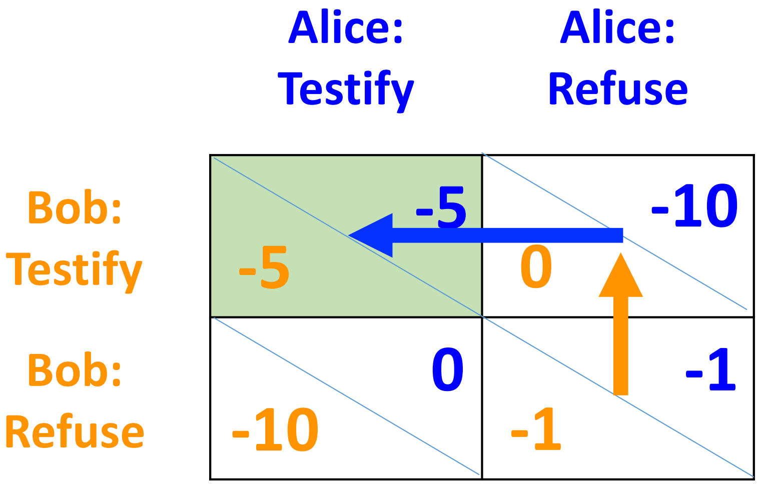
Nash equilibrium

If you knew in advance what your opponent was going to do, what would you do?

- If Bob knew that Alice was going to refuse, then it be rational for Bob to testify (he'd get 0 years, instead of 1).
- If Alice knew that Bob was going to testify, then it would be rational for her to testify (she'd get 5 years, instead of 10).
- If Bob knew that Alice was going to testify, then it would be rational for him to testify (he'd get 5 years, instead of 10).

A ***Nash equilibrium*** is an outcome such that foreknowledge of the other player's action does not cause either player to change their action.



	Alice: Testify	**Alice: Refuse**
Bob: Testify	-5 -5	-10 0
Bob: Refuse	0 -10	-1 -1

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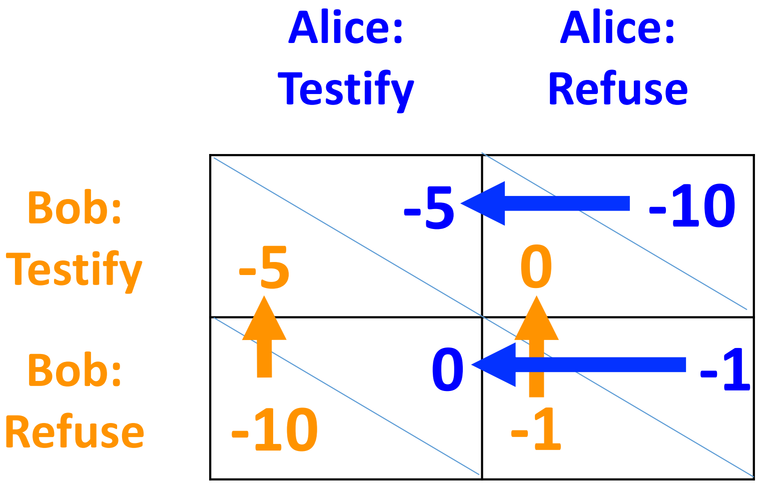
Dominant strategy

If you didn't know in advance what your opponent was going to do, what would you do?

- If Bob knew that Alice was going to refuse, then it be rational for Bob to testify (he'd get 0 years, instead of 1).
- If Bob knew that Alice was going to testify, then it would still be rational for him to testify (he'd get 5 years, instead of 10).

A ***dominant strategy*** is an action that minimizes cost, for one player, regardless of what the other player does.



	Alice: Testify	Alice: Refuse
Bob: Testify	-5 -5	-10 0
Bob: Refuse	0 -10	-1 -1

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What makes it a Prisoner’s Dilemma?

We use that term to mean a game in which

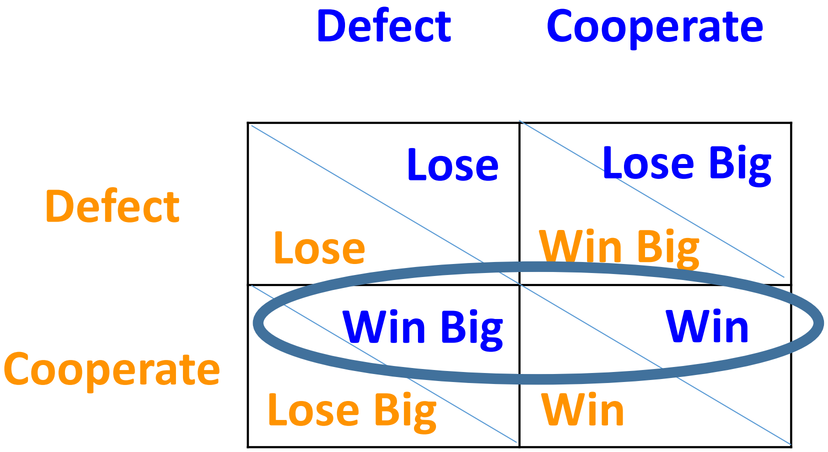
- Defecting is the ***dominant strategy*** for each player, therefore
- (Defect,Defect) is the only ***Nash equilibrium***, even though
- (Defect,Defect) is not a ***Pareto-optimal solution***.

	Defect	**Cooperate**
Defect	**Lose** **Lose**	**Lose Big** **Win Big**
Cooperate	**Win Big** **Lose Big**	**Win** **Win**

http://en.wikipedia.org/wiki/Prisoner’s_dilemma

Prisoner’s Dilemma vs. Stag Hunt

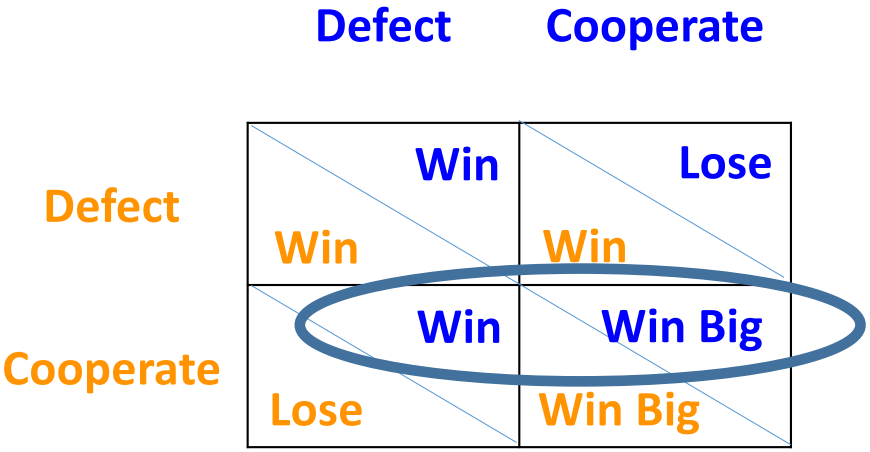
Prisoner’s Dilemma



	Defect	**Cooperate**
Defect	**Lose** **Lose**	**Lose Big** **Win Big**
Cooperate	**Win Big** **Lose Big**	**Win** **Win**

Players ***improve*** their winnings by defecting unilaterally

Stag Hunt



	Defect	**Cooperate**
Defect	**Win** **Win**	**Lose** **Win**
Cooperate	**Win** **Lose**	**Win Big** **Win Big**

Players ***reduce*** their winnings by defecting unilaterally

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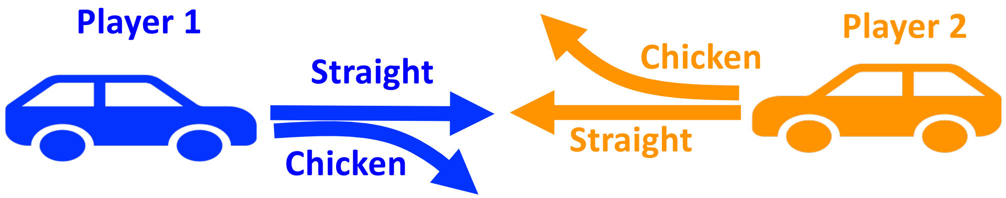
Payoff matrices

- Working for RAND (a defense contractor) in 1950, Flood and Dresher formalized the “Prisoner’s Dilemma” (PD): a class of payoff matrices that encourages betrayal. Was used as a worst-case scenario for the cold war; policies were designed to avoid it.
- Jean-Jacques Rosseau (Swiss philosopher, 1700s) invented the “Stag Hunt” (SH): a class of payoff matrices that reward cooperation, but don’t force it. Has been used as a model of climate-change treaties.
- Both PD and SH have stable Nash equilibria. The “Game of Chicken” is a popular subject in movies (*Rebel Without a Cause*, *Footloose*, *Crazy Rich Asians*) because of its inherent instability: the only way to win is by convincing your opponent to lose.

Game of Chicken



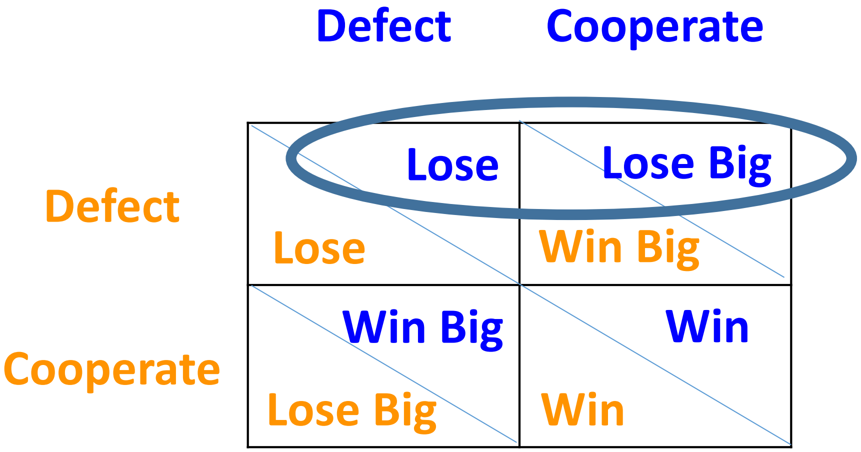
		Alice	
		Straight	Chicken
Bob	Straight	-10 -10	-1 2
	Chicken	2 -1	1 1

- Two players each bet $1000 that the other player will chicken out
- Outcomes:
  - If one player chickens out, he loses $1000, and the other wins $2000
  - If both players chicken out, they each keep their original $1000
  - If neither player chickens out, they both lose $10,000 (the cost of the car)

http://en.wikipedia.org/wiki/Game_of_chicken

Prisoner’s Dilemma vs. Game of Chicken

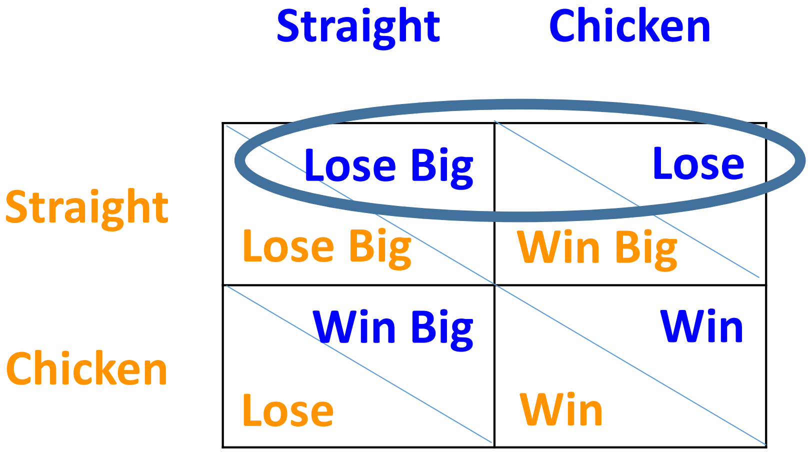
Prisoner’s Dilemma



	Defect	**Cooperate**
Defect	Lose Lose	Lose Big Win Big
Cooperate	Win Big Lose Big	Win Win

Players cut their losses by defecting if the other player defects

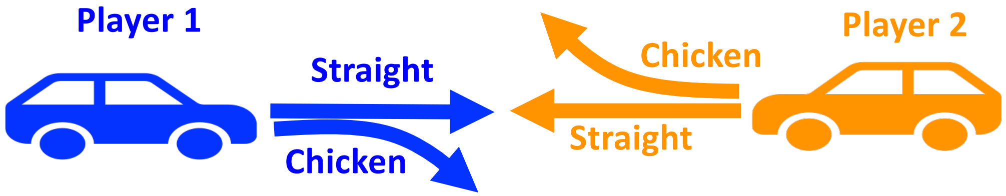
Game of Chicken



	Straight	**Chicken**
Straight	Lose Big Lose Big	Lose Win Big
Chicken	Win Big Lose	Win Win

Defecting, if the other player defects, is the worst thing you can do

Game of Chicken



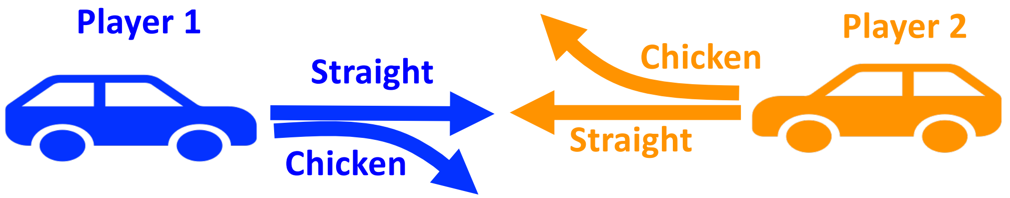
		Alice	
		Straight	Chicken
Bob	Straight	-10 -10	-1 2
	Chicken	2 -1	1 1

- Is there a dominant strategy for either player?
- Is there a Nash equilibrium?

  (straight, chicken) or (chicken, straight)
- *Anti-coordination* game: it is mutually beneficial for the two players to choose different strategies
  - Model of escalated conflict in humans and animals (hawk-dove game)
- How are the players to decide what to do?
  - Bluff! You have to somehow convince your opponent that you will drive straight, no matter what happens, even if it's irrational for you to do so.
  - In that case, the rational thing for your opponent to do is to chicken out.

http://en.wikipedia.org/wiki/Game_of_chicken

Game of Chicken



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  - Model of escalated conflict in humans and animals (hawk-dove game)
- How are the players to decide what to do?
  - Bluff! You have to somehow convince your opponent that you will drive straight, no matter what happens, even if it's irrational for you to do so.
  - In that case, the rational thing for your opponent to do is to chicken out.

Seriously??!! Is there no way to win this game without convincing the other player that you are irrational??!!

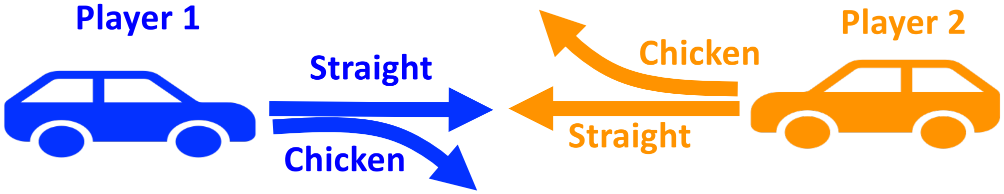
http://en.wikipedia.org/wiki/Game_of_chicken

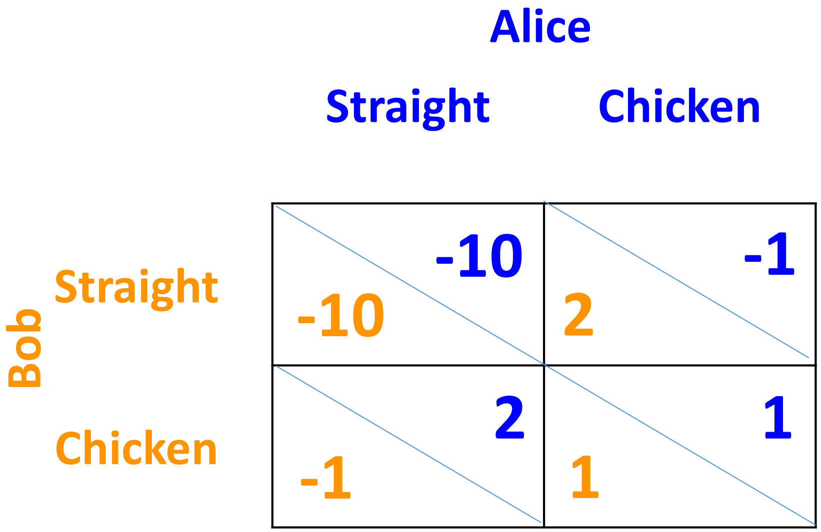
Irrational versus Random

The game of chicken has two different types of Nash equilibria:

- **Be irrational**: Bluff. One player convinces the other that he or she will behave irrationally. The other player concedes the game. Result: (straight,chicken) or (chicken,straight).
- **Be random:** Mixed Nash Equilibrium.
  - Alice chooses a move at random, according to some probability distribution. She tells Bob, in advance, what probability distribution she will use.
  - Bob responds rationally.
  - One of Bob's rational options is to choose his move, also, at random.

Game of Chicken





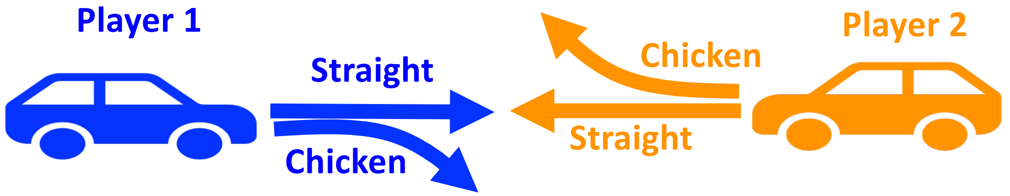
		Alice	
		Straight	Chicken
Bob	Straight	-10 -10	-1 2
	Chicken	2 -1	1 1

- **Mixed strategy:** a player chooses between the different possible actions according to a probability distribution.
- For example, suppose that Bob chooses to behave at random – randomly, every game, he will go straight (s) with probability **1/10**, and chicken out with probability **9/10:**

$$P(B = s) = \frac{1}{10}, \qquad P(B = s) = \frac{9}{10}$$

Can randomness be a ***rational*** action?

Game of Chicken



		Alice	
		Straight	Chicken
Bob	Straight	-10 -10	-1 2
	Chicken	2 -1	1 1

The expected payoff, to Alice, for choosing to go Straight is:

$$E[\text{Payoff}|\text{A} = \text{s}] = \text{P}(\text{B} = s)\, r(s,s) + \text{P}(\text{B} = c)\, \text{r}(s,c) = \left(\frac{1}{10}\right)(-10) + \left(\frac{9}{10}\right)(2) = \frac{8}{10}$$

The expected payoff, to Alice, for choosing to Chicken Out is:

$$E[\text{Payoff}|\text{A} = \text{c}] = \text{P}(\text{B} = s) r(c,s) + \text{P}(\text{B} = c) r(c,c) = \left(\frac{1}{10}\right)(-1) + \left(\frac{9}{10}\right)(1) = \frac{8}{10}$$

Alice has no preference between actions A=S and A=C. Therefore, it is rational for her to choose between the two actions in any arbitrary way, e.g., using a random number generator.

Finding mixed strategy equilibria

		Alice	
		Defect w/ Prob. $1-p$	Coop. w/ Prob. p
Bob	Defect w/ Prob. $1-q$	a w	b x
	Coop. w/ Prob. q	c y	d z

The expected payoff, to **Bob**, for choosing to go Straight is:

$$E[\text{Payoff}|\text{B}=\text{s}]=\text{P}(\text{A}=s)\,r(s,s)+\text{P}(\text{A}=c)\,\text{r}(c,s)=\left(\frac{1}{10}\right)(-10)+\left(\frac{9}{10}\right)(1)=-\frac{1}{10}$$

The expected payoff, to **Bob**, for choosing to Chicken Out is:

$$E[\text{Payoff}|\text{B}=\text{c}]=\text{P}(\text{A}=s)r(s,c)+\text{P}(\text{A}=c)r(c,c)=\left(\frac{1}{10}\right)(-1)+\left(\frac{9}{10}\right)(0)=-\frac{1}{10}$$

So Bob also has no preference between actions B=S and B=C. Therefore, it is rational for him to choose between the two actions in any arbitrary way, e.g., using a random number generator.

Mixed-strategy equilibrium

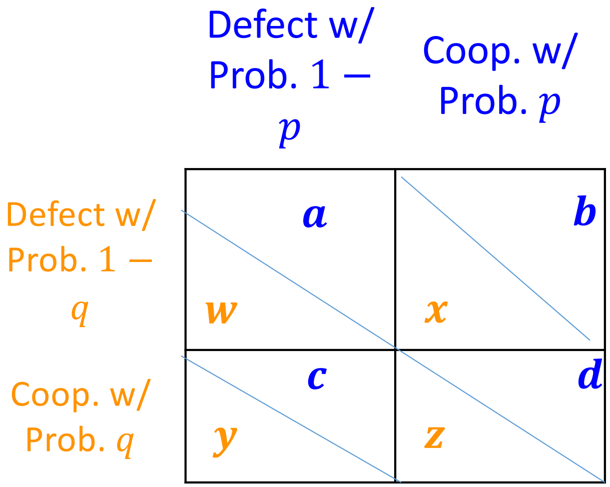
A mixed-strategy equilibrium exists only if there are some $0 \leq p \leq 1$ and $0 \leq q \leq 1$ that solve these equations:

$$(1-p)w + px = (1-p)y + pz$$
$$(1-q)a + qc = (1-q)b + qd$$

If Alice cooperates with probability p, then it is rational for Bob to choose between his two actions at random w/probability q.

If Bob cooperates with probability q, then it is rational for Alice to choose between her two actions at random w/probability p.

This is a mixed strategy equilibrium. It is rational ***<u>on average</u>*** (e.g., if the players will play the same game many times in a row). In any given game play, of course, the outcome could be disastrous for either player or both!



	Defect w/ Prob. $1-p$	Coop. w/ Prob. p
Defect w/ Prob. $1-q$	a w	b x
Coop. w/ Prob. q	c y	d z

Try the quiz!

https://us.prairielearn.com/pl/course_instance/147925/assessment/2405058

Does every game have a mixed-strategy equilibrium?

A mixed-strategy equilibrium exists only if there are some $0 \leq p \leq 1$ and $0 \leq q \leq 1$ that solve these equations:

$$(1-p)w + px = (1-p)y + pz$$
$$(1-q)a + qc = (1-q)b + qd$$

That's not necessarily possible for every game. For example, it's not true for Prisoner's Dilemma.

- Prisoner's Dilemma has only one fixed-strategy Nash equilibrium (both players defect).
- Stag Hunt has two fixed-strategy Nash equilibria (either both players cooperate, or both players defect), and one mixed-strategy equilibrium (each player cooperates with probability 1/10).
- The Game of Chicken has:
  - 2 fixed strategy Nash equilibria (Alice defects while Bob cooperates, or vice versa)
  - 1 mixed-strategy Nash equilibrium (both Alice and Bob each defect with probability 1/10).

	Defect w/ Prob. $1-p$	Coop. w/ Prob. p
Defect w/ Prob. $1-q$	a w	b x
Coop. w/ Prob. q	c y	d z

Existence of Nash equilibria

- Any game with a finite set of actions has at least one Nash equilibrium (which may be a mixed-strategy equilibrium).
- If a player has a dominant strategy, there exists a Nash equilibrium in which the player plays that strategy and the other player plays the *best response* to that strategy.
- If both players have dominant strategies, there exists a Nash equilibrium in which they play those strategies.

Summary

- Dominant strategy
  - a strategy that's optimal for one player, regardless of what the other player does
  - Not all games have dominant strategies
- Nash equilibrium
  - an outcome (one action by each player) such that, knowing the other player's action, each player has no reason to change their own action
  - Every game with a finite set of actions has at least one Nash equilibrium, though it might be a mixed-strategy equilibrium.
- Pareto optimal
  - an outcome such that neither player would be able to win more without simultaneously forcing the other player to lose more
  - Every game has at least one Pareto optimal outcome. Usually there are many, representing different tradeoffs between the two players.
- Mixed strategies
  - A mixed strategy is optimal only if there's no reason to prefer one action over the other, i.e., if $0 \leq p \leq 1$ and $0 \leq q \leq 1$ such that:

$$(1-p)w + px = (1-p)y + pz$$
$$(1-q)a + qc = (1-q)b + qd$$