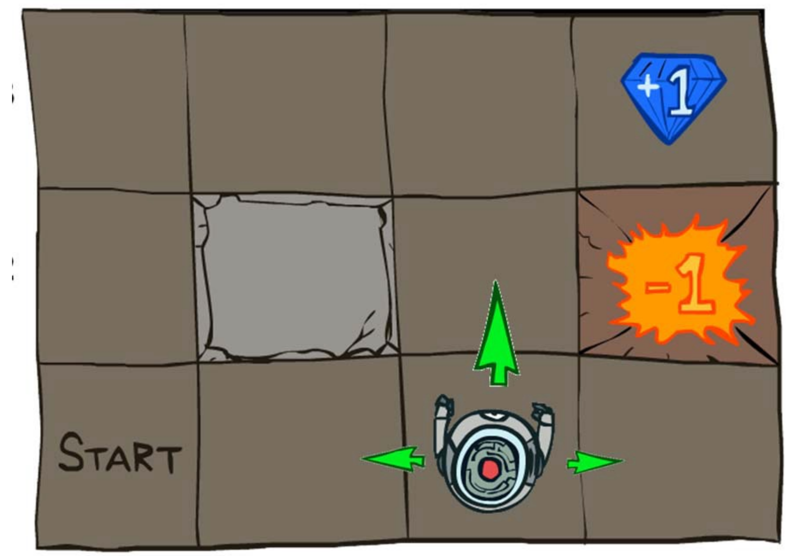
CS440/ECE448 Lecture 20: Markov Decision Processes

Mark Hasegawa-Johnson, 3/2024



Grid World

Invented and drawn by Peter Abbeel and Dan Klein, UC Berkeley CS 188

Outline

- Problem statement
- Utility
- The discount factor
- Value Iteration
- Policy Iteration
- Comparison of value iteration and policy iteration

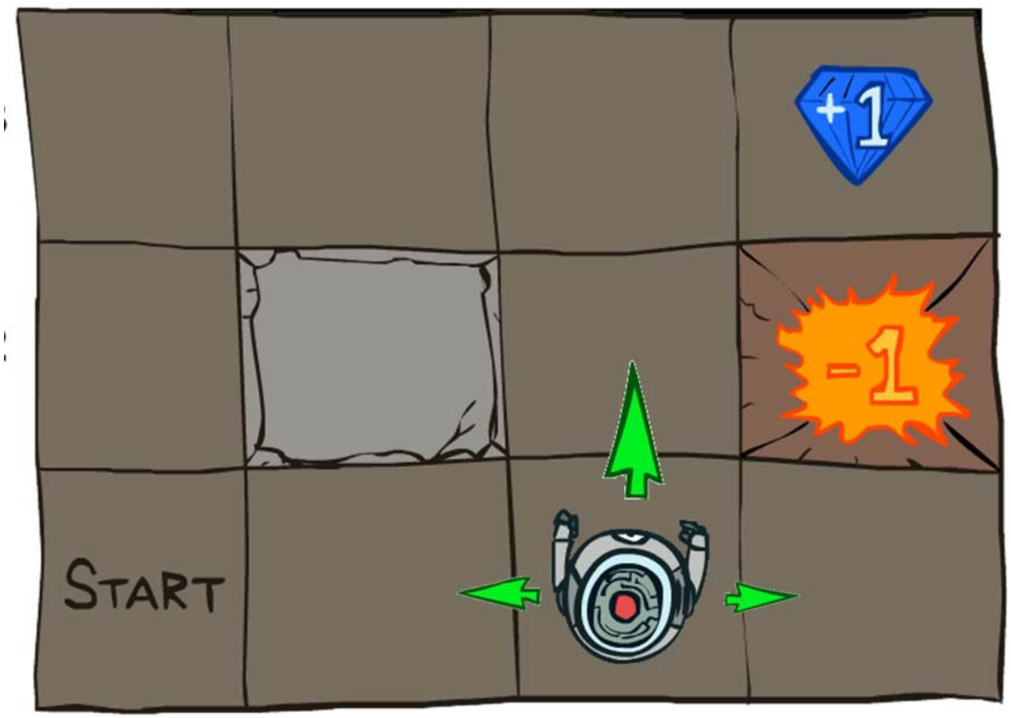
How does an intelligent agent plan its actions?

- If there is no randomness: Use A* search to plan the best path
- What if our movements are affected by randomness?

Example: Grid World

Invented by Peter Abbeel and Dan Klein

- Maze-solving problem: state is $s = (i, j)$, where $0 \le i \le 2$ is the row and $0 \le j \le 3$ is the column.
- The robot is trying to find its way to the diamond.
- If it reaches the diamond, it gets a reward of $r((0,3)) = +1$ and the game ends.
- If it falls in the fire it gets a reward of $r((1,3)) = -1$ and the game ends.

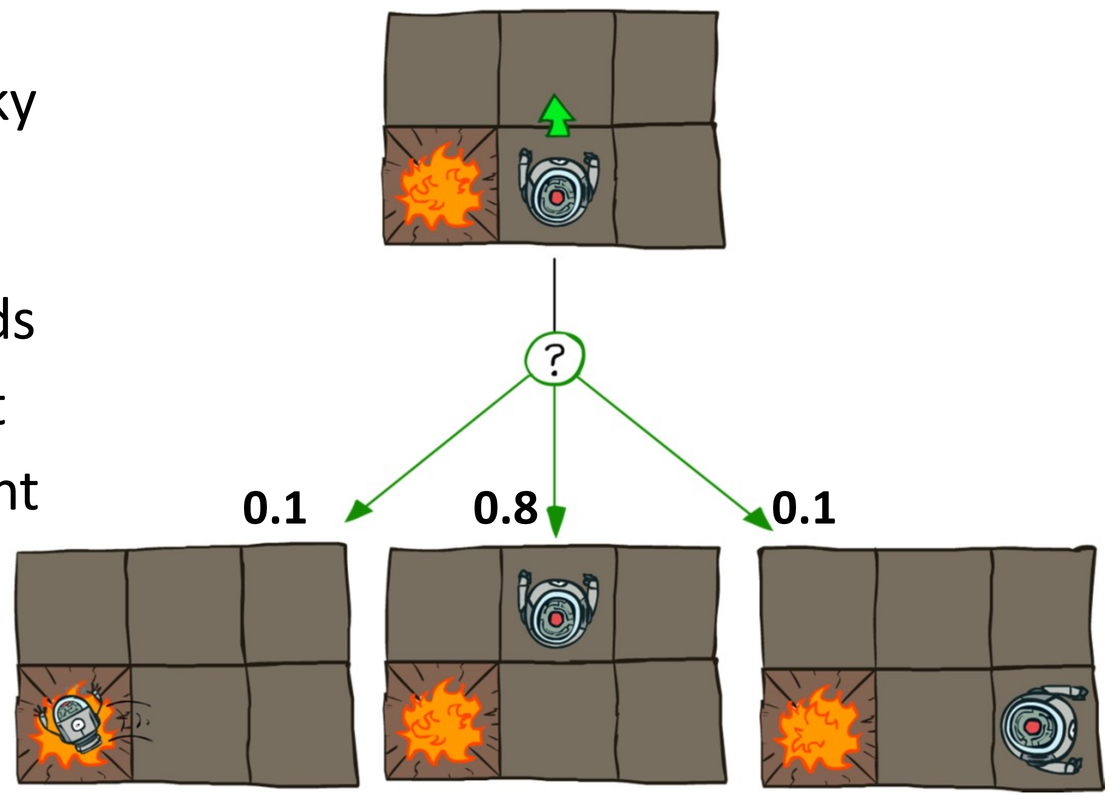


Example: Grid World

Invented by Peter Abbeel and Dan Klein

Randomness: the robot has shaky actuators. If it tries to move forward,

- With probability 0.8, it succeeds
- With probability 0.1, it falls left
- With probability 0.1, it falls right



Markov Decision Process

A Markov Decision Process (MDP) is defined by:

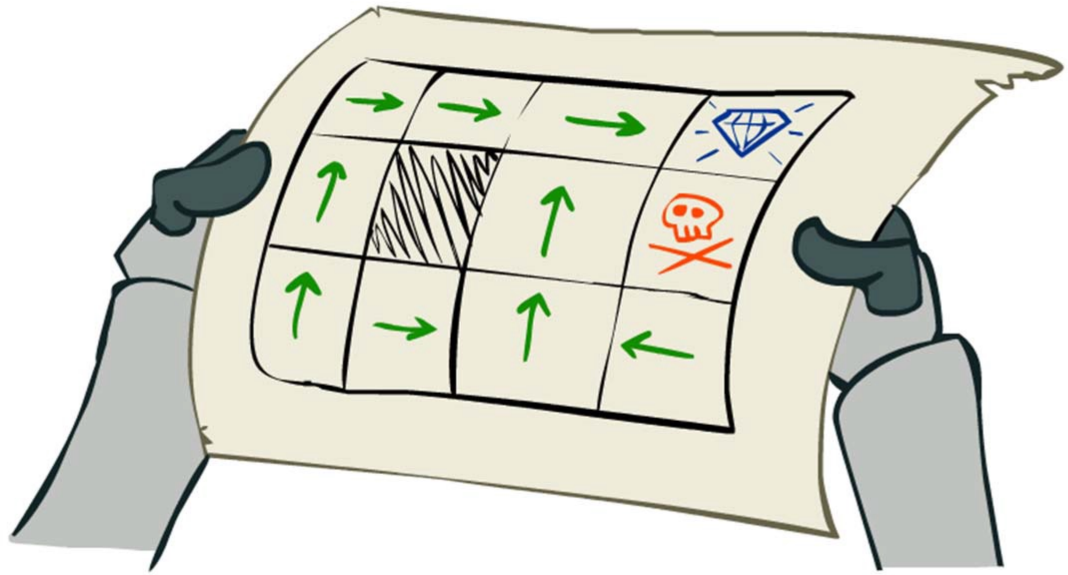
- A set of states, $s \in \mathcal{S}$
- A set of actions, $a \in \mathcal{A}$
- A transition model, $P(S_{t+1} = s_{t+1} | S_t = s_t, a_t)$
  - S_t is the state at time t
  - a_t is the action taken at time t (not random)
- A reward function, $r(s)$

Solving an MDP: The Policy

- The solution to a maze is a path: the shortest path from start to goal
- In MDP, finding 1 path is not enough: randomness might cause us to accidentally deviate from the optimal path.

Solving an MDP: The Policy

- Since $P(S_{t+1} = s_{t+1} | S_t = s_t, a_t)$ and $r(s)$ depend only on the state (the model is Markov), a complete solution can be expressed as follows:
- What is the best action to take in any given state?
- A policy, $a = \pi(s)$, is a function telling you, for any state s, what is the best action to take in that state.



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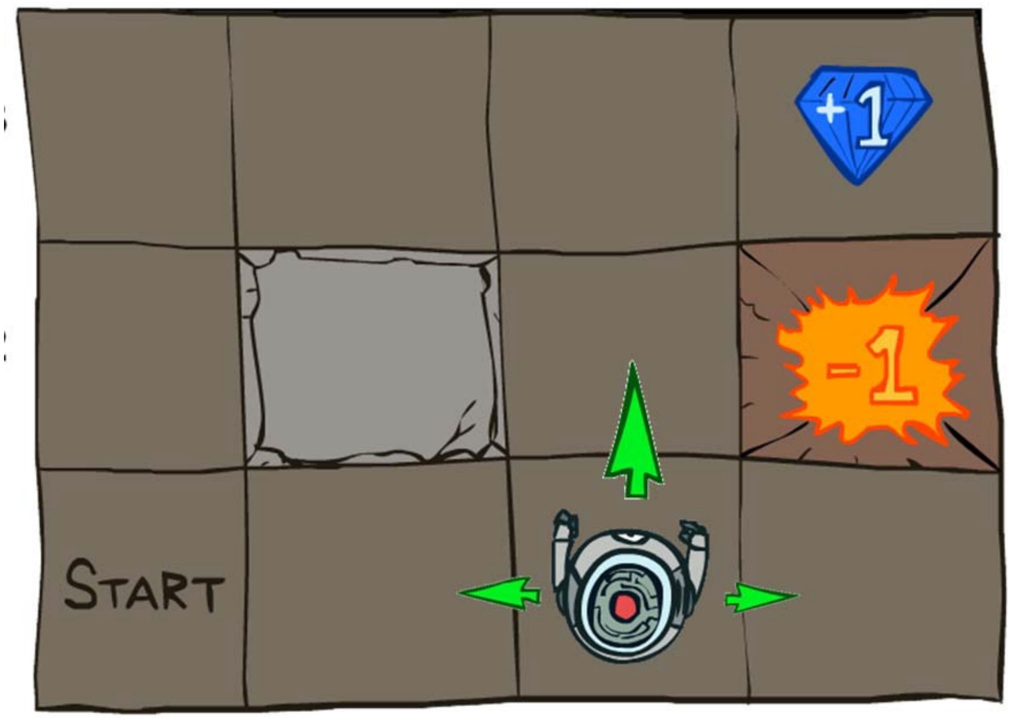
Utility

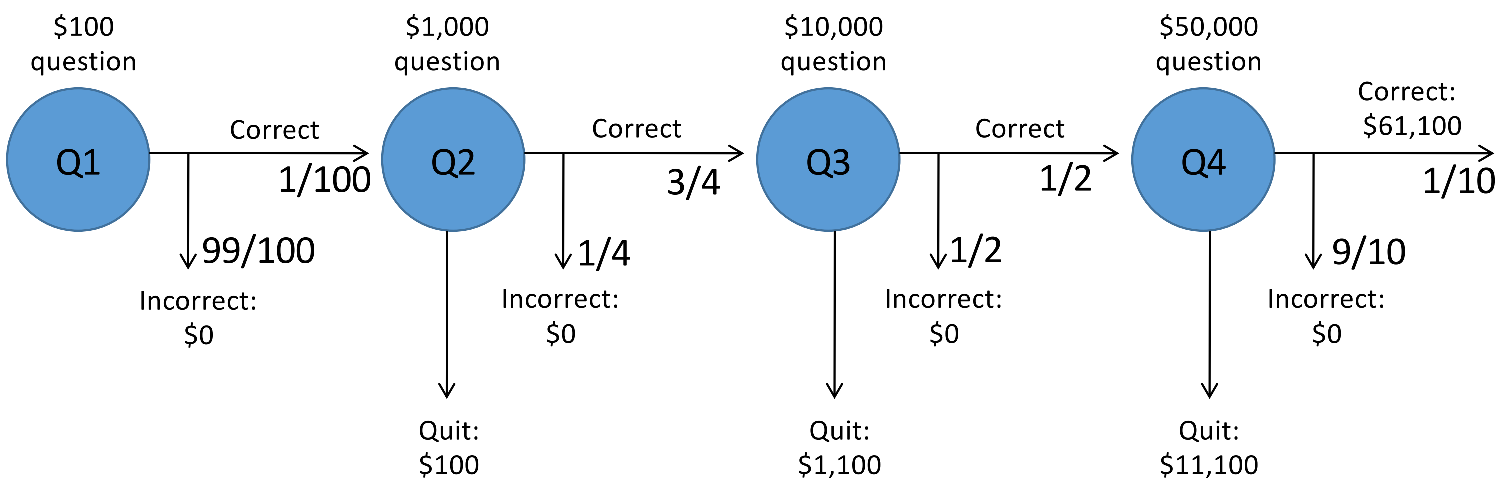
The utility of a state, $u(s)$, is defined to be:

- the sum of all current and future rewards that can be achieved if we start in state s,
- …if we choose the best possible sequence of actions,
- …and if we average over all possible results of those actions.



Example: Game show

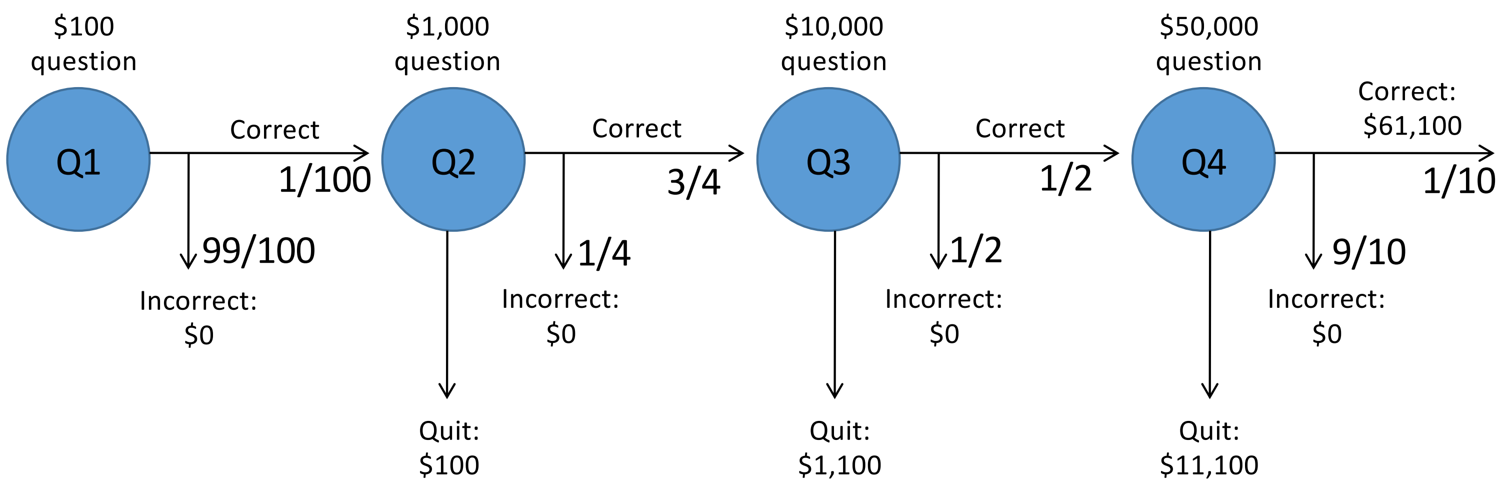
- You've been offered a spot as a contestant in a game show.
- Reward: you receive successively larger prizes for each question you answer correctly, but if you answer any question incorrectly, you lose it all.
- Transition: the questions become harder and harder to answer.
- Actions: after each question, you can decide whether to take another question, or stop.



Example: Game show

Policy:

- If you’ve correctly answered N-1 questions, should you attempt question QN, or stop?

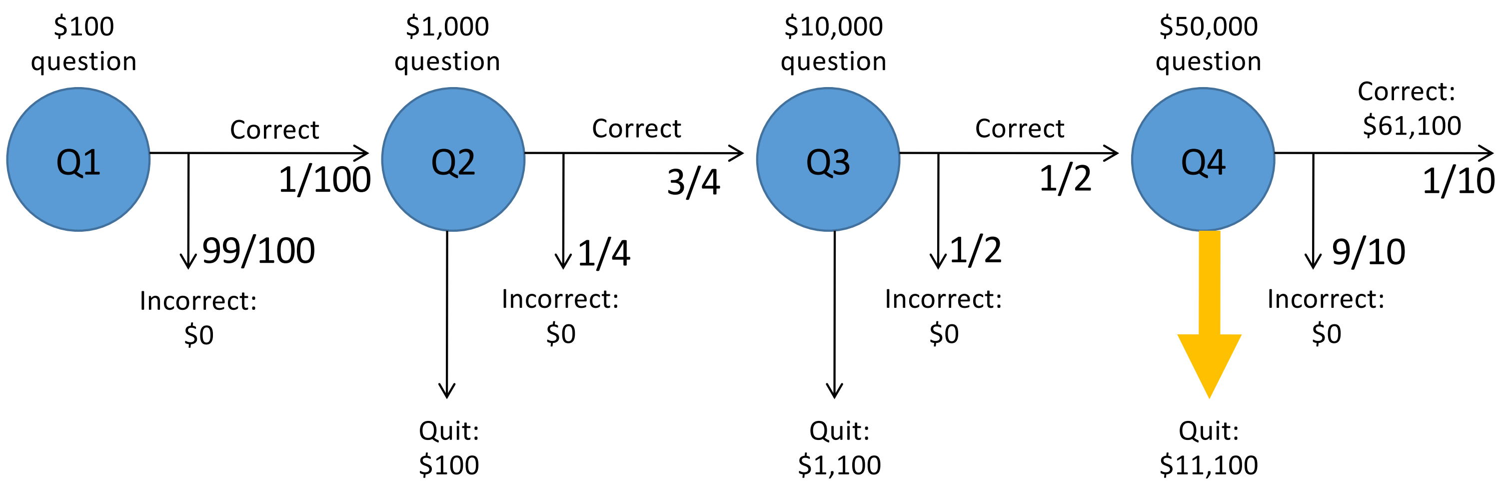


Example: Game show

Policy $\pi(Q4)$: If you've correctly answered 3 questions, should you attempt question Q4, or stop?

- If you stop: total reward is $11,100
- If you attempt Q4: expected total reward is $\frac{1}{10} \times 61100 + \frac{9}{10} \times 0 = \$6110$

Policy: $\pi(Q4) = \text{stop}$. Utility: $u(Q4) = \$11{,}100$

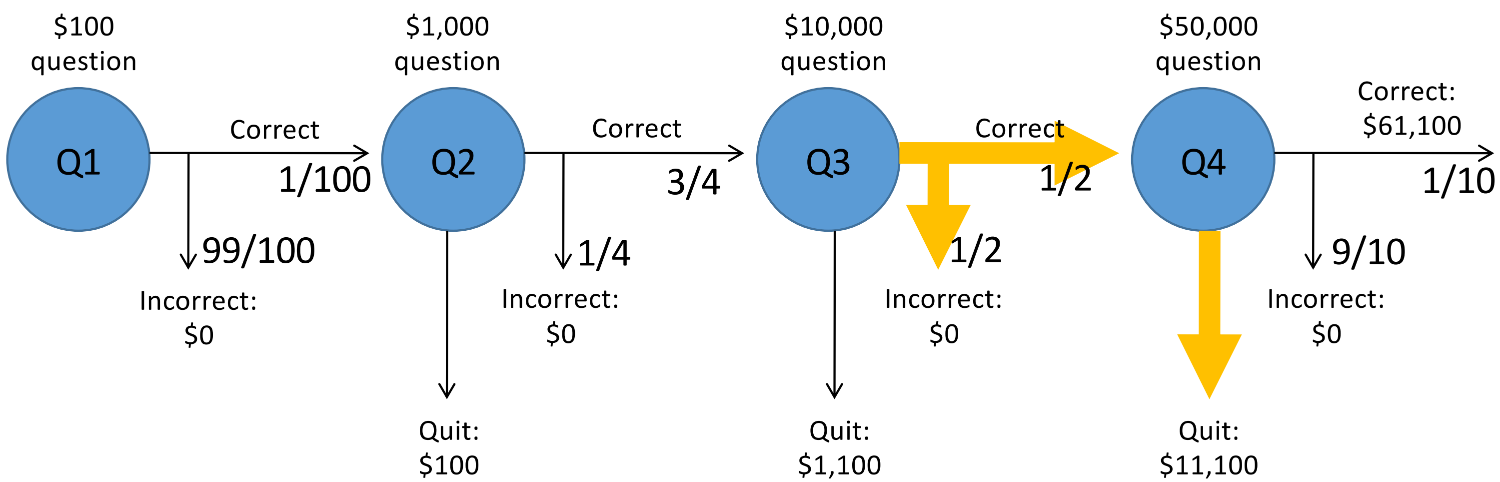


Example: Game show

Policy $\pi(Q3)$: If you've correctly answered 2 questions, should you attempt question Q3, or stop?

- If you stop: total reward is $1,100
- If you attempt Q3: expected total reward is $\frac{1}{2}\times\$11{,}100+\frac{1}{2}\times 0=\$5550$

Policy: $\pi(Q3)=\text{continue.}$ Utility: $u(Q3)=\$5550$

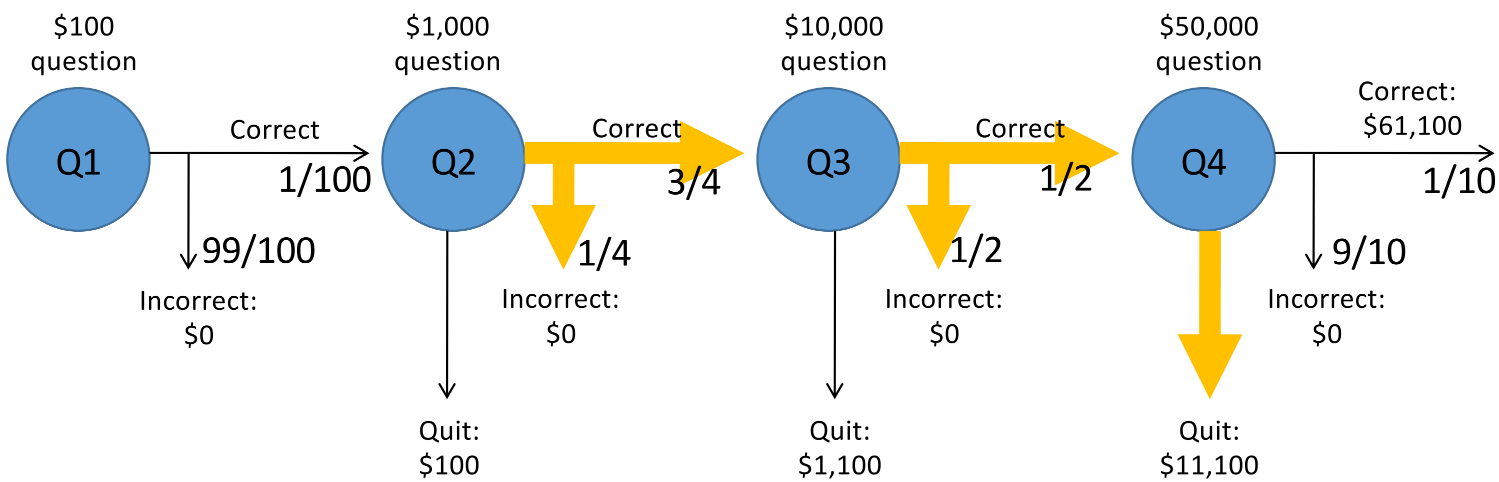


Example: Game show

Policy $\pi(Q2)$: If you've correctly answered 1 question, should you attempt question Q2, or stop?

- If you stop: total reward is \$100
- If you attempt Q2: expected total reward is $\frac{3}{4}\times\$5550 + \frac{1}{4}\times 0 = \$4162.50$

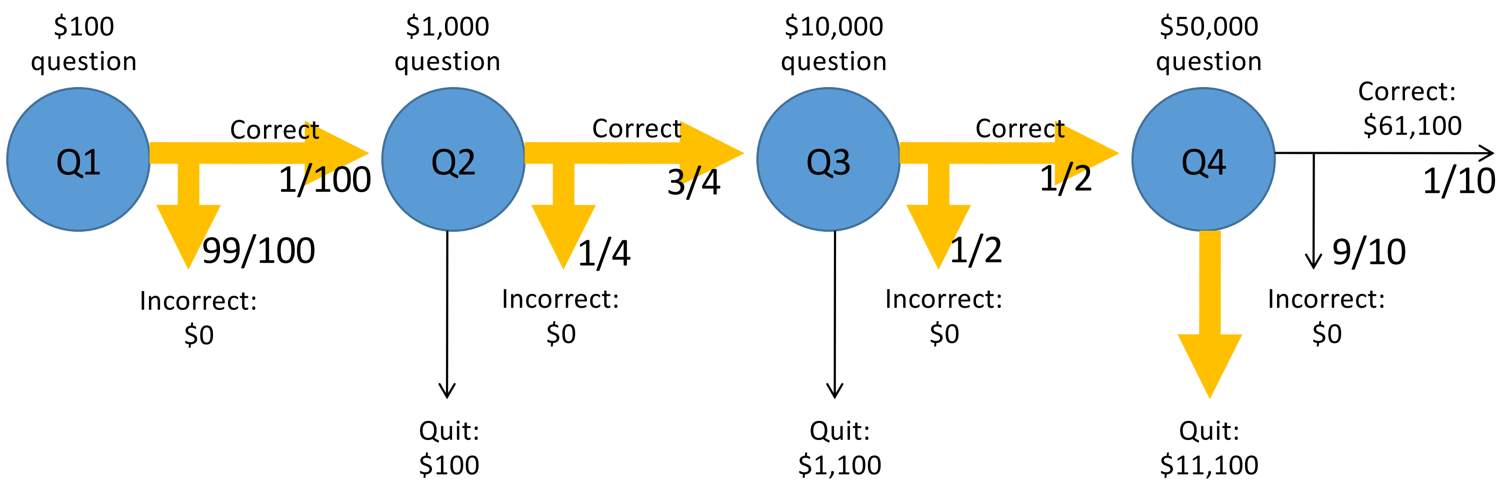
Policy: $\pi(Q2) = \text{continue.}$ Utility: $u(Q2) = \$4162.50$



Example: Game show

Policy $\pi(Q1)$: If you've correctly answered no questions, then you have nothing to lose, so even though the chance of success is very small, you might as well try it!

Policy: $\pi(Q1) = \text{continue.}$ Utility: $u(Q1) = \$41.63$



Utility

The utility of a state, u(s), is

- …the maximum, over all possible sequences of actions, of
- …the expected value, over all possible results of those actions, of
- …the total of all future rewards.

$$u(s) = r(s) + \max_a \sum_{s'} P(s'|s,a)\left(r(s') + \max_{a'} \sum_{s''} P(s''|s',s')(r(s'') + \cdots \cdots \cdots)\right)$$

Utility

The utility of a state, u(s), is

- …the maximum, over all possible sequences of actions, of
- …the expected value, over all possible results of those actions, of
- …the utility of the resulting state.

$$u(s) = r(s) + \max_{a} \sum_{s'} P(S_{t+1} = s' | S_t = s, a) u(s')$$

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Discount factor

You have just won a contest sponsored by the Galaxia Foundation. They offer you the choice of two options:

- $60,000 right now, or…
- $1000 per year, paid to you and your heirs annually forever.

Which option is better?

Discount factor

- Inflation has averaged 3.8% annually from 1960 to 2024.
- Equivalently, \$1000 received one year from now is worth approximately \$962 today.
- A reward of \$1000 annually forever (starting today, t=0) is equivalent to an immediate reward of

$$r = \sum_{t=0}^{\infty} 1000(0.962)^t = \frac{1000}{1 - 0.962} = \$26{,}316$$

We call the factor $\gamma = 0.962$ the discount factor.

Discount factor

Why is a dollar tomorrow worth less than a dollar today?

- A dollar will buy less tomorrow
- The person paying you might go out of business
- You might have to go into hiding and become unable to collect

The discount factor, γ , is our model of the unknowable uncertainty of promised future rewards.

Public domain image of J. Wellington Wimpy, the character who popularized the saying “I will gladly pay you Tuesday for a hamburger today.”

https://commons.wikimedia.org/wiki/File:Wimpyhotdog.png

The Bellman Equation

$$u(s) = r(s) + \gamma \max_a \sum_{s'} P(S_{t+1} = s' | S_t = s, a) u(s')$$

- The Bellman equation specifies the utility of the current state.
- In solving the Bellman equation, we also find the optimum action, which is the policy.
- However…

The Bellman Equation

$$\begin{bmatrix} u(1) \\ \vdots \\ u(n) \end{bmatrix} = \begin{bmatrix} r(1) \\ \vdots \\ r(n) \end{bmatrix} + \gamma \max_a \begin{bmatrix} P(1|1,a) & \cdots & P(1|n,a) \\ \vdots & \ddots & \vdots \\ P(n|1,a) & \cdots & P(n|n,a) \end{bmatrix} \begin{bmatrix} u(1) \\ \vdots \\ u(N) \end{bmatrix}$$

- If there are n states, then the Bellman equation is n nonlinear equations in n unknowns.
- There is no closed-form solution; we must use an iterative solution

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Value iteration

The Bellman Equation:

$$u(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) u(s')$$

Value iteration solves the Bellman equation iteratively. In iteration number i, for $i = 0,1, \ldots,$

- For all states s, $u_i(s)$ is an estimate of $u(s)$
- Start out with $u_0(s) = 0$ for all states
- In the i^{th} iteration,

$$u_i(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) u_{i-1}(s')$$

Value iteration

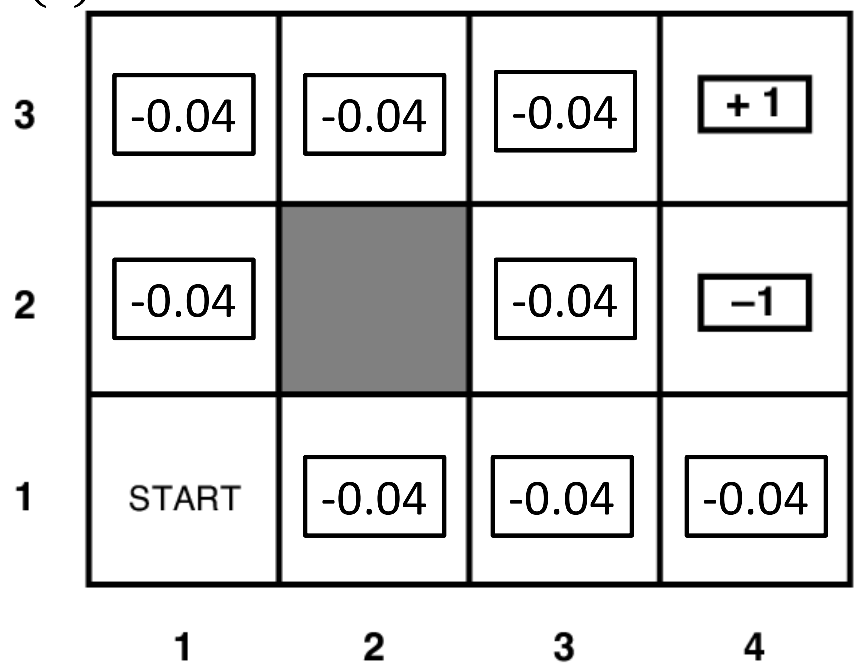
$$u_i(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) u_{i-1}(s')$$

Notice that:

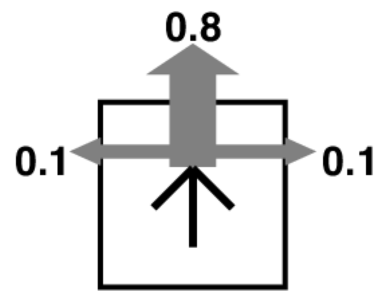
- After i iterations, $u_i(s)$ has information about the rewards earned in the first i steps after the agent starts the maze
- A policy designed based on $u_i(s)$ will act in order to maximize reward in the first i steps of the maze
- In this sense, it's kind of like BFS: each iteration explores farther and farther away from the starting state.

Example: Grid world

$r(s)$



Transition model $P(s'|s,a)$:

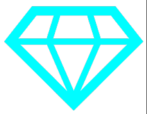


Assume a "loitering penalty" of $r(s) = -0.04$ for all non-terminal states.

Value Iteration: Iteration 1

$$u_1(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a)u_0(s')$$

$u_1(s)$

-0.04	-0.04	-0.04	
-0.04		-0.04	
-0.04	-0.04	-0.04	-0.04

$r(s)$

-0.04	-0.04	-0.04	
-0.04		-0.04	
-0.04	-0.04	-0.04	-0.04

$u_0(s)$

0	0	0	
0		0	
0	0	0	0

Value Iteration: Iteration 2

$$u_2(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a)u_1(s')$$

$u_2(s)$

−0.08	−0.08	+0.75	
−0.08		−0.08	
−0.08	−0.08	−0.08	−0.08

=

$r(s)$

−0.04	−0.04	−0.04	
−0.04		−0.04	
−0.04	−0.04	−0.04	−0.04

$+\gamma \max_a$

$\sum_{s'} P(s'|s,\text{down})u_1(s')$

−0.04	−0.04	+0.06	
−0.04		−0.14	
−0.04	−0.04	−0.04	−0.04

$\sum_{s'} P(s'|s,\text{up})u_1(s')$

−0.04	−0.04	+0.06	
−0.04		−0.14	
−0.04	−0.04	−0.04	−0.81

$\sum_{s'} P(s'|s,\text{left})u_1(s')$

−0.04	−0.04	−0.04	
−0.04		−0.04	
−0.04	−0.04	−0.04	−0.14

$\sum_{s'} P(s'|s,\text{right})u_1(s')$

−0.04	−0.04	+0.79	
−0.04		−0.81	
−0.04	−0.04	−0.04	−0.14

Quiz

Try the quiz!

https://us.prairielearn.com/pl/course_instance/147925/assessment/2403836

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Method 2: Policy Iteration

- **Policy Evaluation:** $u_i(s) = r(s) + \gamma \sum_{s'} P(s'|s, \pi_i(s)) u_i(s')$
  - Given a **fixed** policy $\pi_i(s)$,
  - Calculate the resulting utility $u_i(s)$.
- **Policy Improvement:** $\pi_{i+1}(s) = \underset{a}{\operatorname{argmax}} \sum_{s'} P(s'|s, a) u_i(s')$
  - Given a **fixed** utility $u_i(s)$,
  - Find an improved $\pi_{i+1}(s)$.

- Unlike Value Iteration, this is guaranteed to converge in a finite number of steps (less than or equal to the number of distinct policies)

Step 1: Policy Evaluation

Bellman equation: n nonlinear equations in n unknowns:

$$\begin{bmatrix} u(1) \\ \vdots \\ u(n) \end{bmatrix} = \begin{bmatrix} r(1) \\ \vdots \\ r(n) \end{bmatrix} + \gamma \max_a \begin{bmatrix} P(1|1,a) & \cdots & P(1|n,a) \\ \vdots & \ddots & \vdots \\ P(n|1,a) & \cdots & P(n|n,a) \end{bmatrix} \begin{bmatrix} u(1) \\ \vdots \\ u(N) \end{bmatrix}$$

Policy Evaluation: n linear equations in n unknowns:

$$\begin{bmatrix} u_i(1) \\ \vdots \\ u_i(n) \end{bmatrix} = \begin{bmatrix} r(1) \\ \vdots \\ r(n) \end{bmatrix} + \gamma \begin{bmatrix} P(1|1,\pi_i(1)) & \cdots & P(1|n,\pi_i(n)) \\ \vdots & \ddots & \vdots \\ P(n|1,\pi_i(1)) & \cdots & P(n|n,\pi_i(n)) \end{bmatrix} \begin{bmatrix} u_i(1) \\ \vdots \\ u_i(N) \end{bmatrix}$$

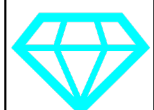
The difference is that policy evaluation is linear, so it can be solved by inverting a matrix: $\boldsymbol{u}_i = (\boldsymbol{I} - \gamma \boldsymbol{P}_i)^{-1}\boldsymbol{r}$.

Example: Grid World

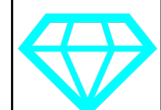
Policy Evaluation: $u_i(s) = r(s) + \gamma \sum_{s'} P(s'|s, \pi_i(s))u_i(s')$

- Assume the initial policy is $\pi_1(s) =$ "Go Right" for all states
- Solve the linear equations to find $u_1(s)$

$u_1(s)$

$+0.50$	$+0.69$	$+0.74$	
-0.65		-0.90	
-1.40	-1.44	-1.39	-1.40

$\pi_1(s)$

$\rightarrow$	$\rightarrow$	$\rightarrow$	
$\rightarrow$		$\rightarrow$	
$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$

Policy Improvement

Policy Evaluation: $u_i(s) = r(s) + \gamma \sum_{s'} P(s'|s, \pi_i(s)) u_i(s')$

Policy Improvement: $\pi_{i+1}(s) = \underset{a}{\text{argmax}} \sum_{s'} P(s'|s, a) u_i(s')$

$\pi_2(s)$

→	→	→	
↑		↑	
↑	→	↑	↑

$u_1(s)$

+0.50	+0.69	+0.74	
−0.65		−0.90	
−1.40	−1.44	−1.39	−1.40

$\pi_1(s)$

→	→	→	
→		→	
→	→	→	→

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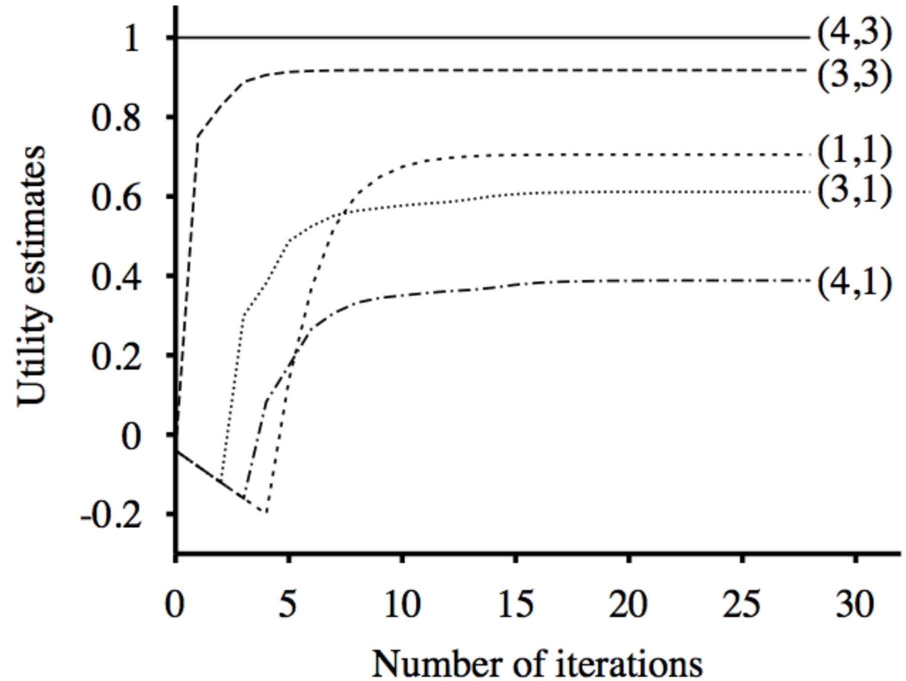
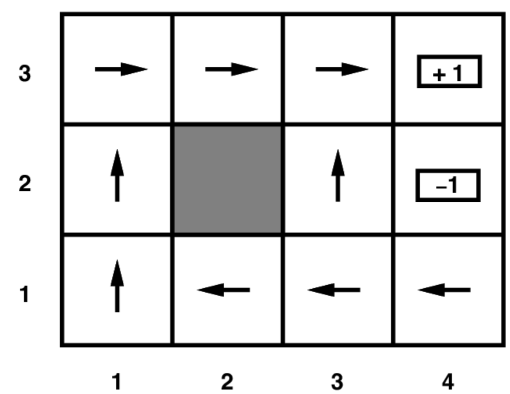
Value iteration

Optimal utilities with discount factor 1
(Result of value iteration)

	1	2	3	4
3	0.812	0.868	0.918	+ 1
2	0.762		0.660	–1
1	0.705	0.655	0.611	0.388



Final policy



Comparison of value iteration and policy iteration

- Bellman equation is n equations in n unknowns; cannot be solved in closed form, needs an iterative solution
- Value iteration
  - Behaves like BFS: each iteration looks one step farther from the start node
  - Usually converges exponentially fast to the correct policy
  - However, if there are loops possible in the maze, may never converge exactly
- Policy iteration
  - Kind of like gradient descent: evaluate a policy, then improve it
  - Guaranteed to converge in a finite number of steps
  - Harder to implement, and might take a while before it starts to converge

Summary

- Bellman equation:

$$u(s) = r(s) + \gamma \max_a \sum_{s'} P(S_{t+1} = s' | S_t = s, a) u(s')$$

- Value iteration:

$$u_i(s) = r(s) + \gamma \max_a \sum_{s'} P(S_{t+1} = s' | S_t = s, a) u_{i-1}(s')$$

- Policy iteration:

$$u_i(s) = r(s) + \gamma \sum_{s'} P(S_{t+1} = s' | S_t = s, \pi_i(s)) u_i(s')$$

$$\pi_{i+1}(s) = \operatorname{argmax}_a \sum_{s'} P(S_{t+1} = s' | S_t = s, a) u_i(s')$$