CS440/ECE 448, Lecture 18: Constraint Satisfaction Problems

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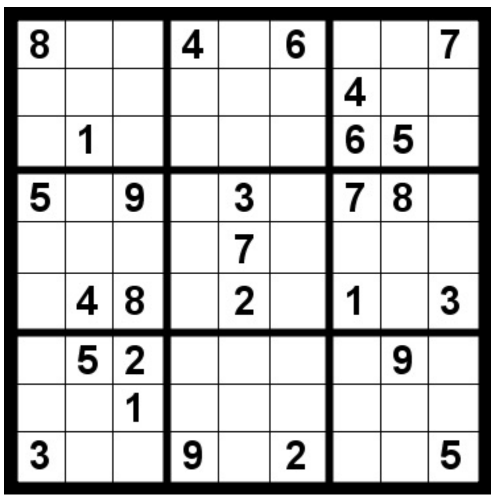
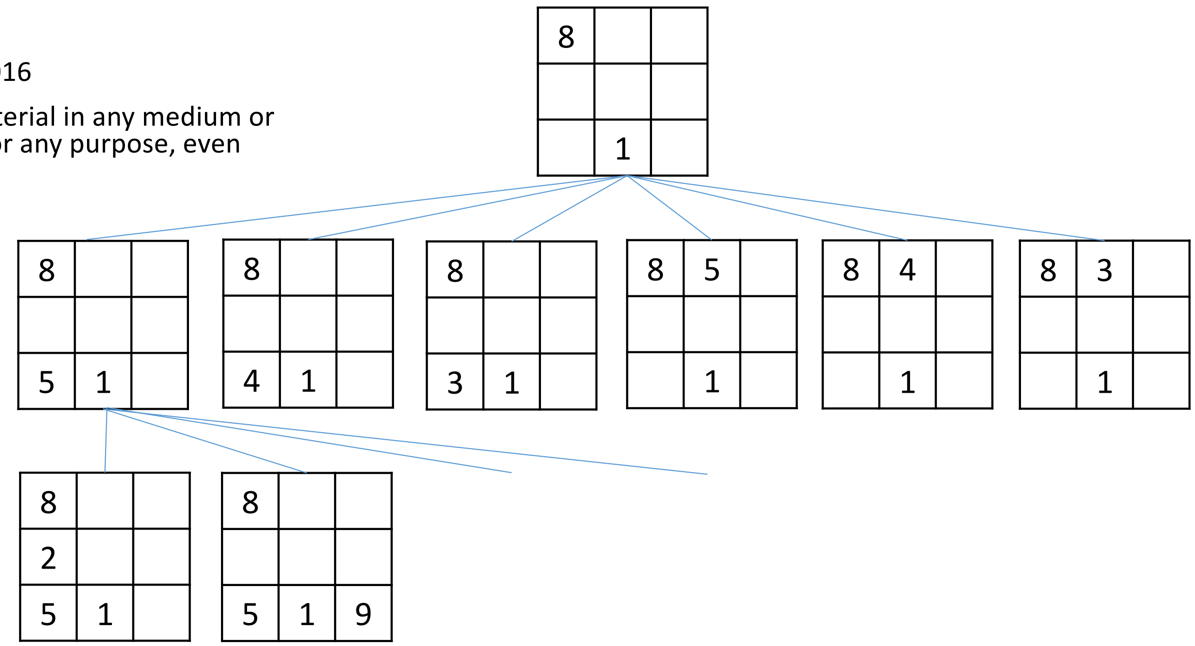
Including some slides written by Svetlana Lazebnik, 9/2016





Content

- What is a CSP? Why is it search? Why is it special?
- Backtracking Search
- $O\{1\}$ heuristics to improve backtracking search
- $O\{N\}$ and $O\{N^2\}$ heuristics: early detection of failure

What is search for?

- Assumptions: single agent, deterministic, fully observable, discrete environment
- **Search for *planning***
  - The path to the goal is the important thing
  - Paths have various costs, depths
- **Search for *assignment***
  - Assign values to variables while respecting certain constraints
  - The goal (complete, consistent assignment) is the important thing

8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
3			9		2			5

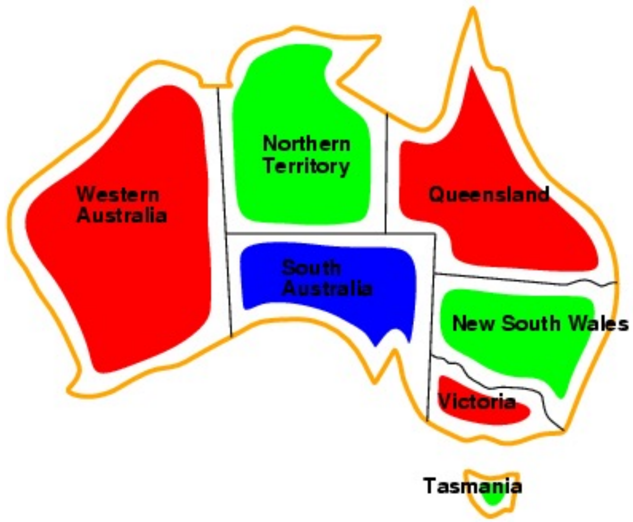
Why are Constraint satisfaction problems (CSPs) just a special case of generic search problems?

- **State** is defined by N variables, each takes one of D possible values
- **Action** = assign a value to a variable
- **Transition Model** = if you have n variables assigned, then assign one more, now you have n+1 variables assigned.
- **Goal test** is a set of constraints specifying allowable combinations of values for subsets of variables.
- **Solution** is a complete, consistent assignment

Example: Map Coloring

- **Variables:** WA, NT, Q, NSW, V, SA, T
- **Domains:** {red, green, blue}
- **Constraints:** adjacent regions must have different colors
  - Logical representation: WA ≠ NT
  - Set representation: (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

Example: Map Coloring



- **Solutions** are *complete* and *consistent* assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Why are Constraint satisfaction problems (CSPs) <u>different from</u> generic search problems?

- Because every path has N steps! So the computational cost of DFS, is the SAME as the cost of BFS, $\boldsymbol{O\{b^m\} = O\{b^d\} = O\{D^N\}}$
  - Path length is N, because there are N variables to assign
  - Branching factor is D, because there are D possible values.
- Meanwhile, space is still a problem. DFS allows us to delete the part of the tree corresponding to an unsuccessful path. So DFS is more useful than BFS.
- Topic of today: how do we use heuristics with DFS?
- Hint: it's not as elegant as A*. There is no provable optimality. In fact…

Computational complexity of CSPs

- The satisfiability (SAT) problem:
  - Given a Boolean formula, is there an assignment of the variables that makes it evaluate to true?

$$(X_1 \vee \overline{X}_7 \vee X_{13}) \wedge (\overline{X}_2 \vee X_{12} \vee X_{25}) \wedge \ldots$$

- SAT and CSP are *NP-complete*
  - NP: a class of decision problems for which
    - the "yes" answer can be verified in polynomial time
    - no known algorithm can find a "yes" answer, from scratch, in polynomial time
  - An NP-complete problem is in NP and every other problem in NP can be efficiently reduced to it (Cook, 1971)
  - Other NP-complete problems: graph coloring, n-puzzle, generalized sudoku
  - It is not known whether P = NP, i.e., no efficient algorithms for solving SAT in general are known

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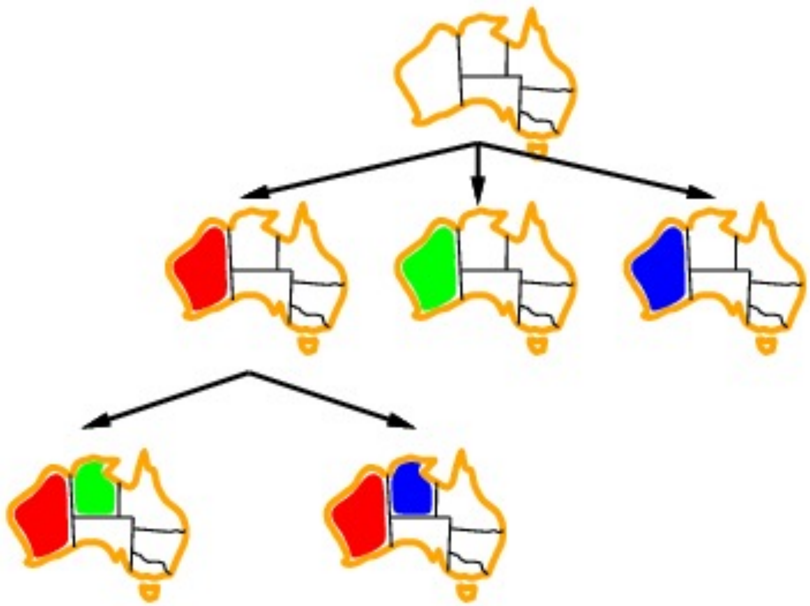
Backtracking search

- In CSP's, variable assignments are **commutative**
  - For example, *[WA = red then NT = green]* is the same as *[NT = green then WA = red]*
- We only need to consider assignments to a single variable at each level (i.e., we fix the order of assignments)
  - There are N! different orderings of the variables. If we choose a particular ordering, and then never change it, we reduce computational complexity by a factor of N!
  - With a fixed order, there are still D^N possible paths.
  - At each level, choose one of the D possible assignments, and explore to see if it gives you a solution. If not, **backtrack**: delete the whole sub-tree (that's why we're using DFS!), and try something different.
- Depth-first search for CSPs with single-variable assignments is called **backtracking search**

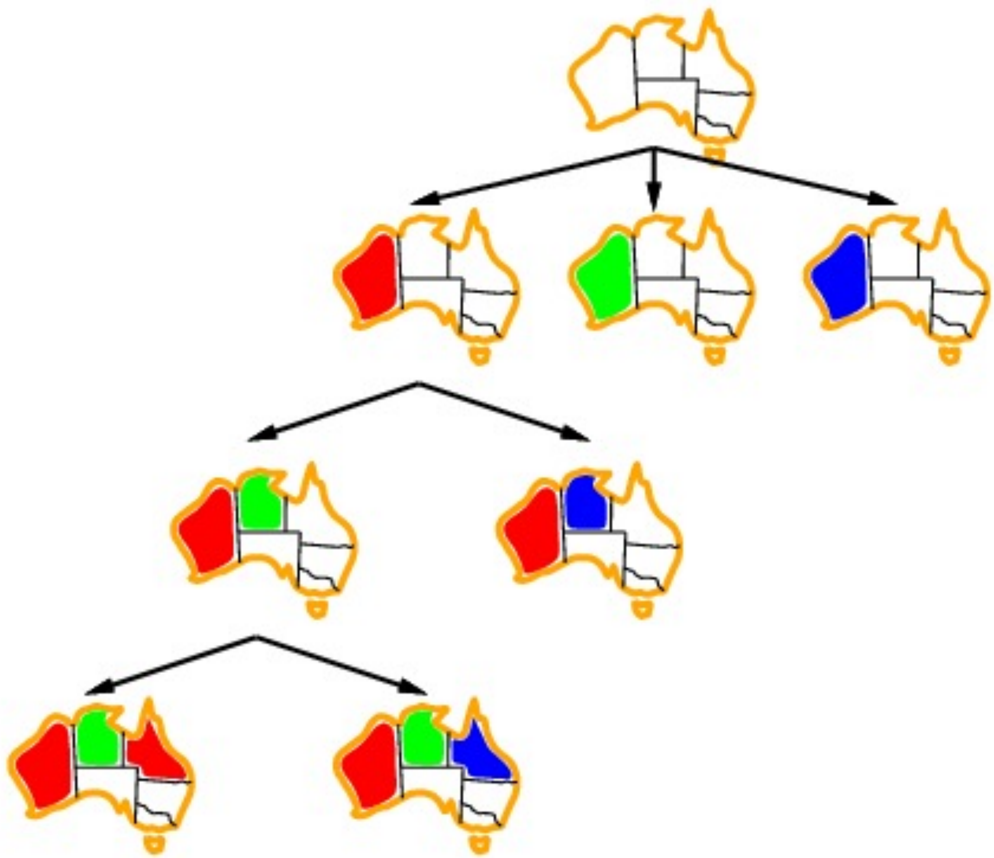
Example

Example

Example



Example



Backtracking search algorithm

function RECURSIVE-BACKTRACKING(*assignment*, *csp*)
  if *assignment* is complete **then return** *assignment*
  $\boxed{var \leftarrow \text{SELECT-UNASSIGNED-VARIABLE}(\text{VARIABLES}[csp], assignment, csp)}$
  for each *value* **in** $\boxed{\text{ORDER-DOMAIN-VALUES}(var, assignment, csp)}$
    if *value* is consistent with *assignment* given CONSTRAINTS[*csp*]
      add $\{var = value\}$ to *assignment*
      $result \leftarrow$ RECURSIVE-BACKTRACKING(*assignment*, *csp*)
      if $result \neq failure$ **then return** *result*
      remove $\{var = value\}$ from *assignment*
  return *failure*

- Making backtracking search efficient:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

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- What is a CSP? Why is it search? Why is it special?
- Backtracking Search
- $O\{1\}$ heuristics to improve backtracking search
  1. Given a particular variable, which value should you assign?
  2. Which variable should you consider next?
- $O\{N\}$ and $O\{N^2\}$ heuristics: early detection of failure

Given a variable, in which order should its values be tried?

```
function RECURSIVE-BACKTRACKING(assignment, csp)
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
        if value is consistent with assignment given CONSTRAINTS[csp]
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

- Making backtracking search efficient:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Given a variable, in which order should its values be tried?

- **Least Constraining Value (LCV) Heurstic**:
  - First assignment to try: the value that rules out the fewest future possibilities

- Key intuition: maximize the probability of success.

Given a variable, in which order should its values be tried?

- **Least Constraining Value (LCV) Heurstic**:
  - Try the following assignment first: to the variable you're studying, the value that rules out the fewest values in the remaining variables

Which assignment for Q should we choose?

Which variable should be assigned next?

function RECURSIVE-BACKTRACKING(*assignment, csp*)
  if *assignment* is complete **then return** *assignment*
  $\boxed{var \leftarrow \text{SELECT-UNASSIGNED-VARIABLE}(\text{VARIABLES}[csp], assignment, csp)}$
  for each *value* **in** $\boxed{\text{ORDER-DOMAIN-VALUES}(var, assignment, csp)}$
    if *value* is consistent with *assignment* given CONSTRAINTS[*csp*]
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- Making backtracking search efficient:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Which variable should be assigned next?

- **Key intuitions:**
  - If there is a solution possible, it will still be possible, regardless of the order in which you study the variables.
  - So choosing a VARIABLE is easier than choosing a VALUE. Just minimize the branching factor.

- **Least Remaining Values (LRV) Heuristic:**
  - Choose the variable with the fewest legal values
- **Most Constraining Variable (MCV) Heuristic:**
  - Choose the variable that imposes the most constraints on the remaining variables

Which variable should be assigned next?

- **Least Remaining Values (LRV) Heuristic:**
  - Choose the variable with the fewest legal values.
  - At this stage, we should have chosen SA, not Q, because SA has only one legal value.

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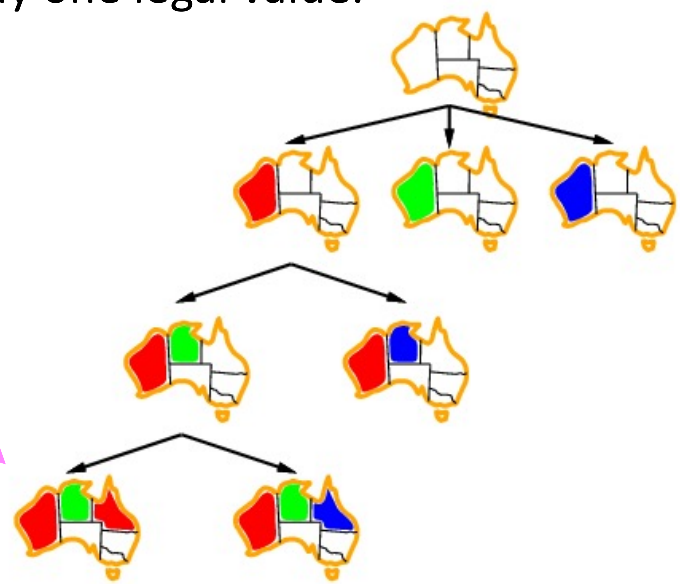
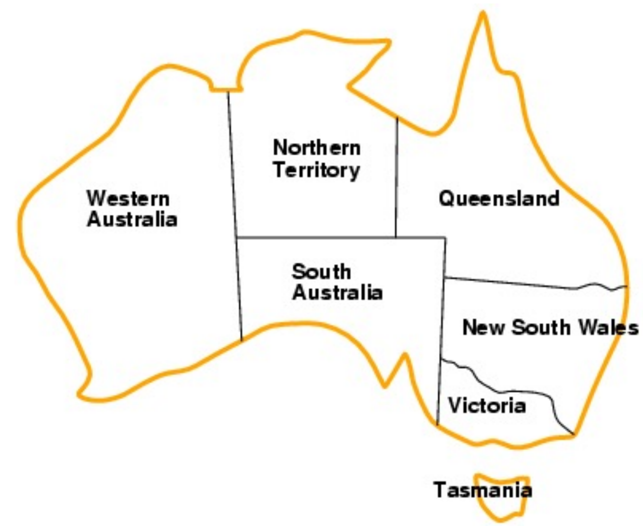


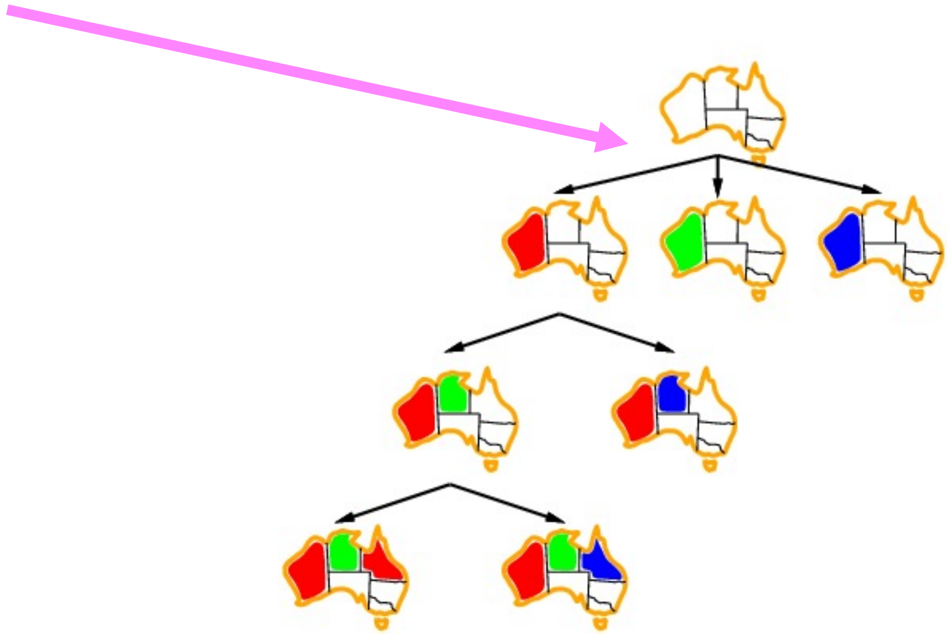


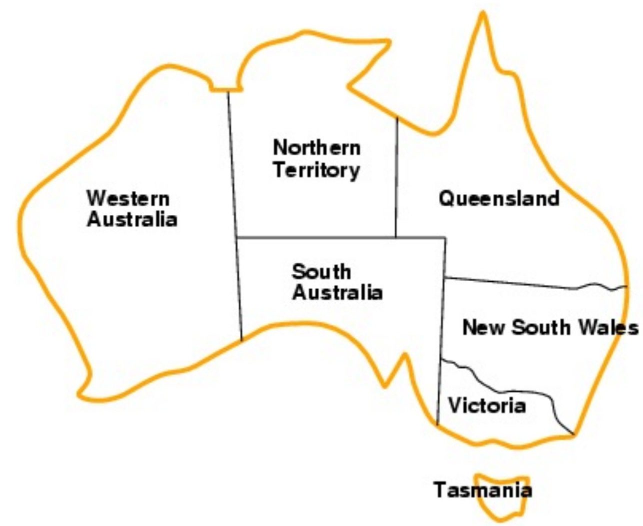
Which variable should be assigned next?

- **Most Constraining Variable (MCV) Heuristic:**
  - Choose the variable that imposes the most constraints on the remaining variables
  - Tie-breaker among variables that have equal numbers of LRV

Which variable should be assigned next?

- **Most Constraining Variable (MCV) Heuristic:**
  - Choose the variable that imposes the most constraints on the remaining variables
  - The very first assignment should have been SA (not WA), because it imposes constraints on 5 other variables (WA constrains only 2).





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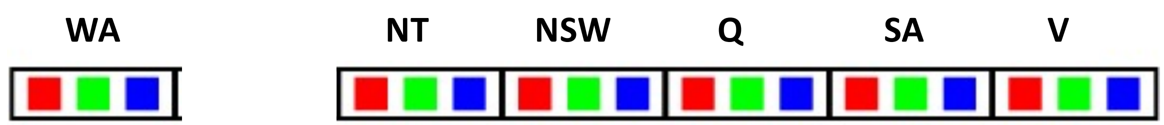
Early detection of failure: O{N} checking

- **Forward Checking:**
  - Check to make sure that every variable still has at least one possible assignment

Early detection of failure: O{N} checking
Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



Early detection of failure: O{N} checking
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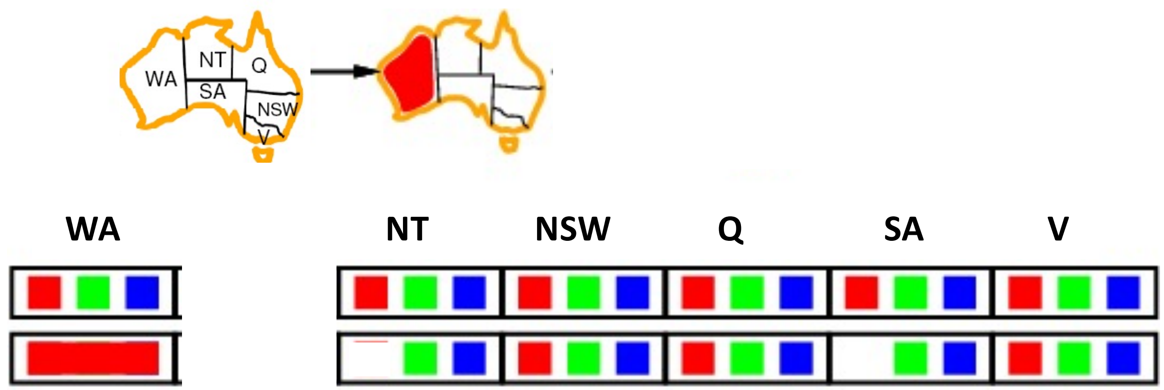

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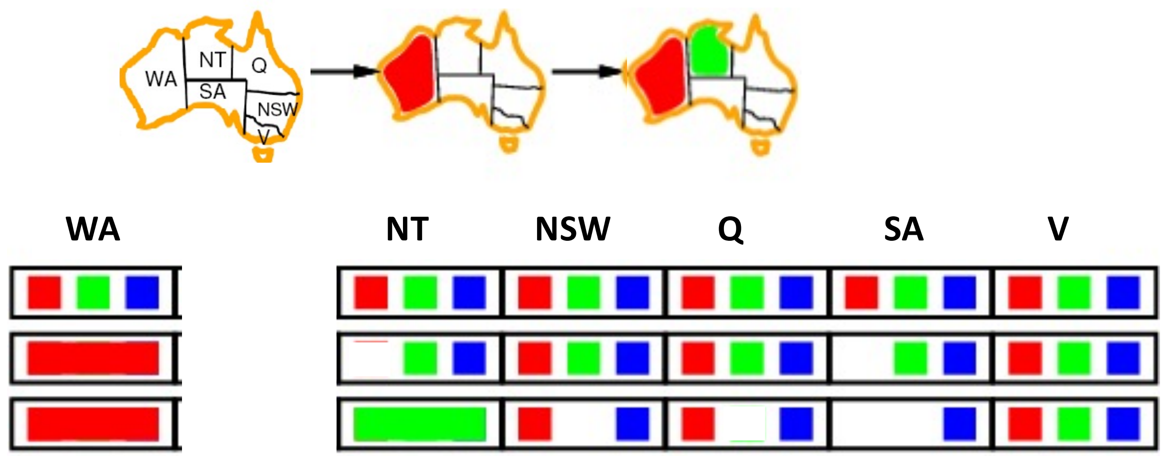

Early detection of failure: O{N} checking Forward checking

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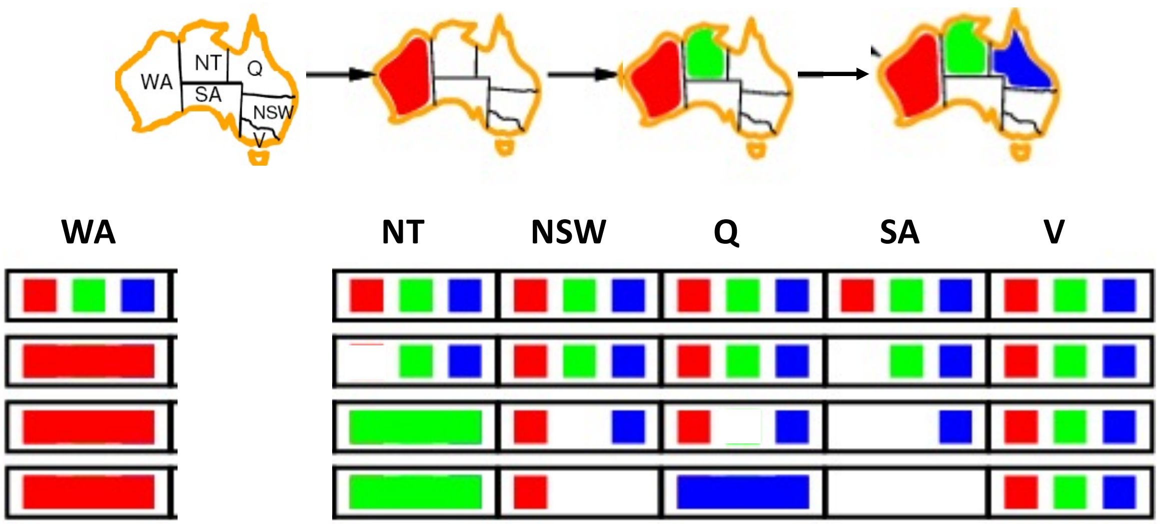


Early detection of failure: O{N^2} checking

- **Arc consistency:**
  - Check to make sure that every PAIR of variables (every “arc”) still has a pair-wise assignment that satisfies all constraints

Arc consistency

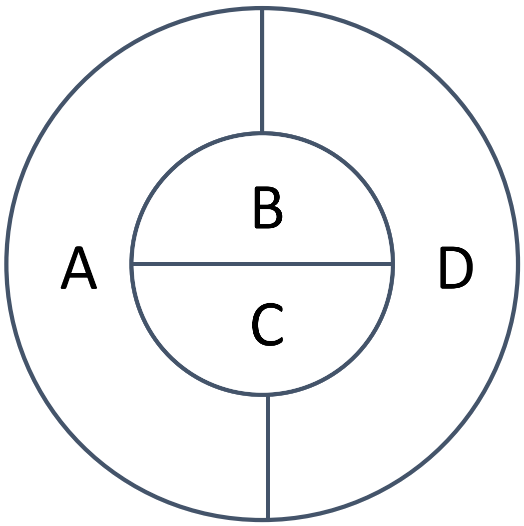
Pair of variables	# remaining assignments
NT, Q	4
NT, SA	2
SA, Q	4
SA, NSW	4
Q, NSW	6
NSW, V	6
SA, V	6

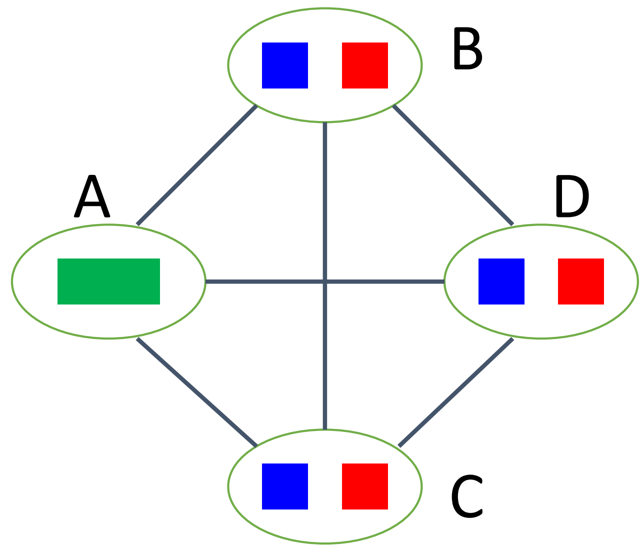
Pair of variables	# remaining assignments
SA, Q	1
SA, NSW	2
Q, NSW	4
NSW, V	6
SA, V	2

Early detection of failure: O{N^2} checking

- **Arc consistency:**
  - If we try an assignment and discover that it leaves any pair of variables with 0 valid assignments, then STOP, and backtrack.
  - Allows us to quit the search process 2 levels earlier than otherwise

Does arc consistency always detect the lack of a solution?





- There exist stronger notions of consistency (path consistency, k-consistency) that trade off complexity of the heuristic versus depth of the search.

Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- **Backtracking** = depth-first search where successor states are generated by considering assignments to a single variable
  - **Variable ordering** (LRV, MCV) and **value selection** (LCV) heuristics can help significantly
  - **Forward checking** says: don't consider an assignment if it leaves any variable with no remaining possible values
  - **Arc consistency** says: don't consider an assignment if it leaves any **pair of variables** with no remaining mutually compatible pair of values
- Complexity of CSPs
  - NP-complete in general (exponential worst-case running time)
  - Typical run-time can be reduced substantially using polynomial-complexity forward-checking and arc-consistency heuristics