Lecture 17: The “animal kingdom” of heuristics: Admissible, Consistent, Zero, Relaxed, Dominant

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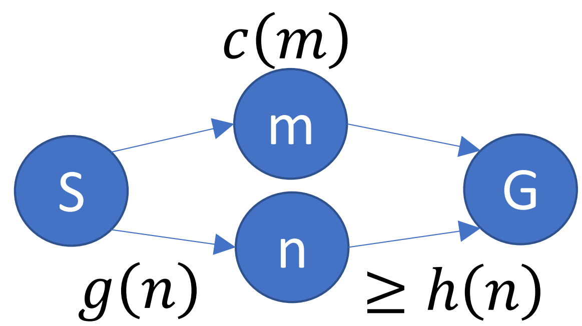
Title image: Peaceable Kingdom by Edward Hicks, National Gallery of Art, Washington, DC

Outline of lecture

1. Admissible heuristics
2. Consistent heuristics
3. The zero heuristic: Uniform Cost Search
4. Relaxed heuristics
5. Dominant heuristics

A* Search

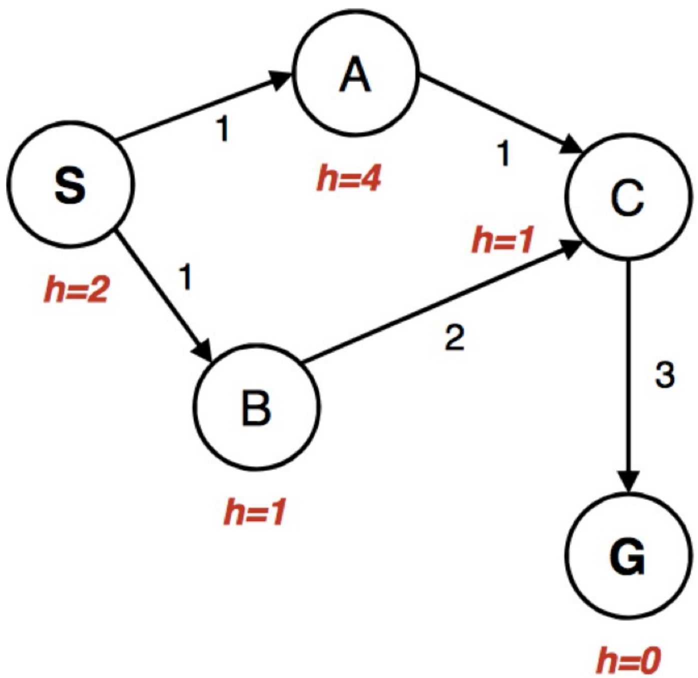


Definition: A* SEARCH

- If $h(n)$ is **admissible** $(d(n) \geq h(n))$, and
- if the frontier is a priority queue sorted according to $g(n) + h(n)$, then
- the FIRST path to goal uncovered by the tree search, path m, is guaranteed to be the SHORTEST path to goal

  ($h(n) + g(n) \geq c(m)$ for every node n that is not on path m)

Bad interaction between A* and the explored set



Explored Set:

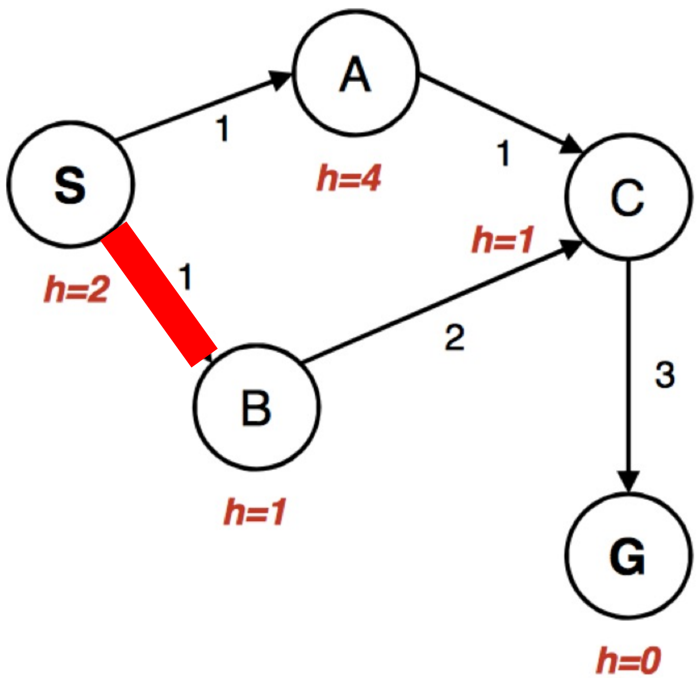
empty

Frontier

S: g(S)+h(S)=2, g(S)=0, parent=none

Expand: S, put its children A and B on the frontier.

Bad interaction between A* and the explored set



Explored Set:

S

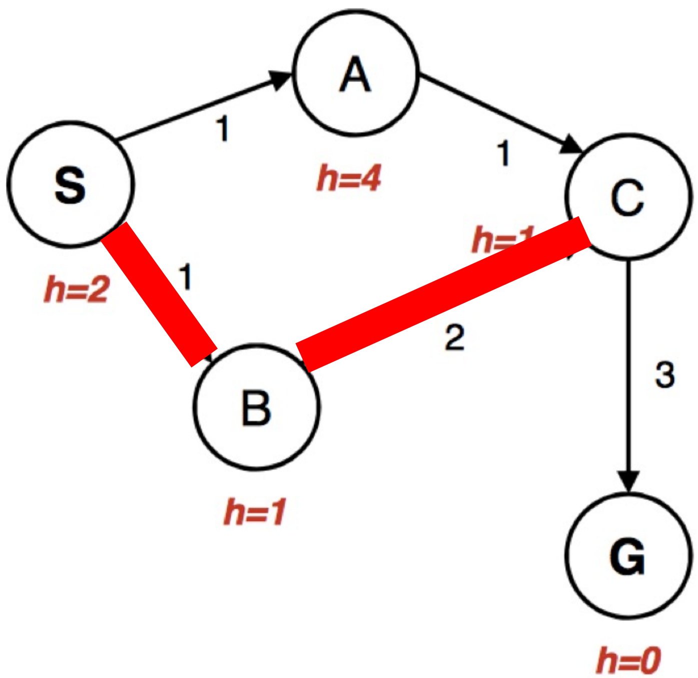
Frontier

A: g(A)+h(A)=5, g(A)=1, parent=S

B: g(B)+h(B)=2, g(B)=1, parent=S

Expand: B, put its child C on the frontier.

Bad interaction between A* and the explored set



Explored Set:

S, B

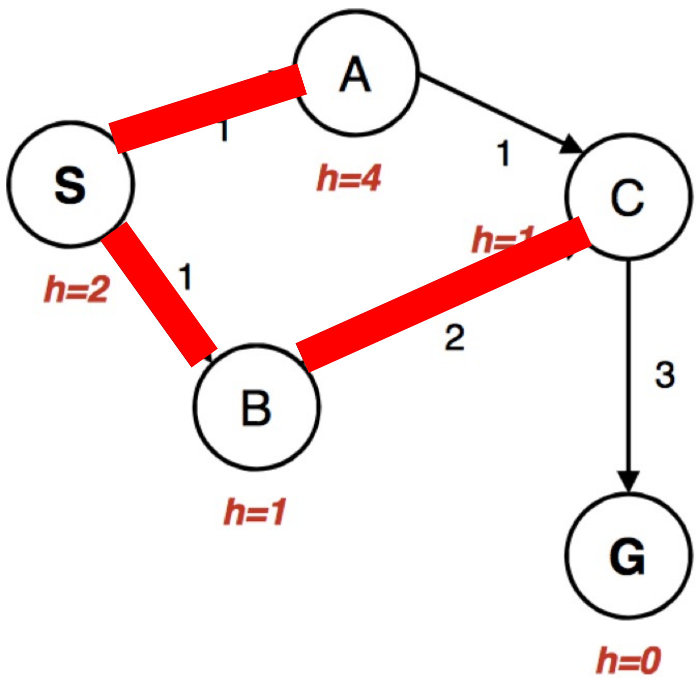
Frontier

A: g(A)+h(A)=5, g(A)=1, parent=S

C: g(C)+h(C)=4, g(C)=3, parent=B

Expand: C, put its child G on the frontier.

Bad interaction between A* and the explored set



Explored Set:

S, B, C

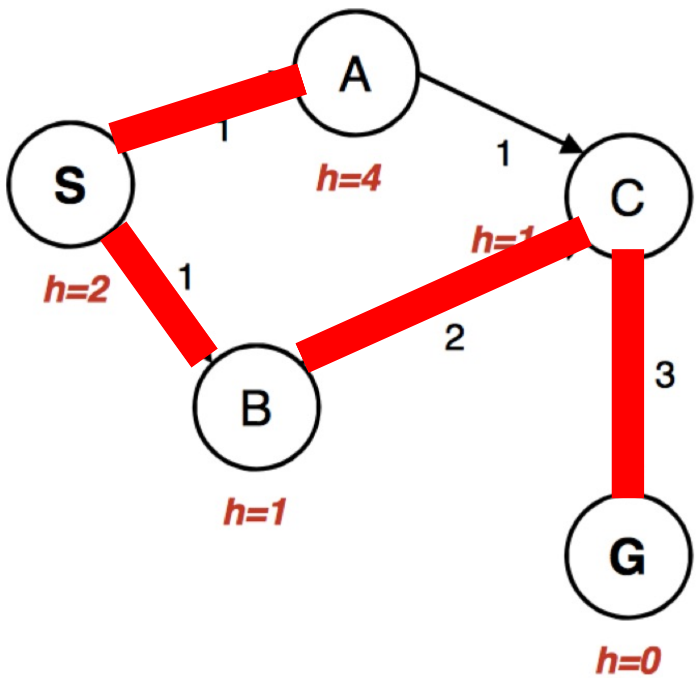
Frontier

A: g(A)+h(A)=5, g(A)=1, parent=S

G: g(G)+h(G)=6, g(G)=6, parent=C

Expand: A. But we can't put its child, C, on the frontier, because C is already in the explored set!

Bad interaction between A* and the explored set



Explored Set:

S, B, C

Frontier

G: g(G)+h(G)=6, g(G)=6, parent=C

Expand: G. Return the path SBCG, with cost 6. OOPS!

Why did this happen?

- Well, because we used an <u>**explored set**</u> instead of an <u>**explored dict**</u>.
  - <u>**explored dict**</u> lists the h(n)+g(n) for each explored state
  - If the same state shows up later, with lower h(n)+g(n), then put it back on the frontier.
  - An explored set undermines A*, but an explored dict works just fine.
- But actually, why did the higher-cost path <u>**SBC**</u> get explored before the lower-cost path <u>**SAC**</u>?
  - That never happens for goal. An <u>**admissible**</u> heuristic guarantees that the first time you pop Goal from the frontier, it will have its lowest cost.
  - Can we make the same idea true for ***every state***, not just the ***goal state?***

Outline of lecture

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2. Consistent heuristics
3. The zero heuristic: Uniform Cost Search
4. Relaxed heuristics
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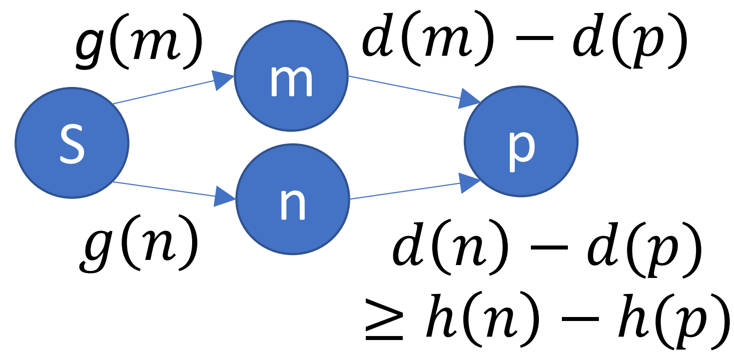
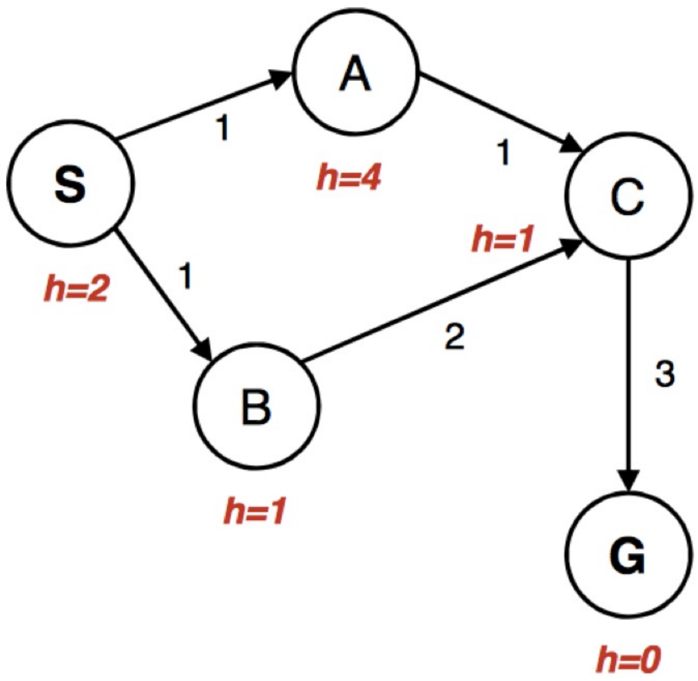
Consistent (monotonic) heuristic



<u>Definition:</u> A **<u>consistent heuristic</u>** is one for which, for every pair of nodesn and p such that $d(n) \geq d(p)$, $d(n) - d(p) \geq h(n) - h(p)$.

In words: the <u>distance between any pair of nodes</u> is **<u>greater than or equal to</u>** the <u>difference in their heuristics</u>.

A* with an inconsistent heuristic



Inconsistent because:

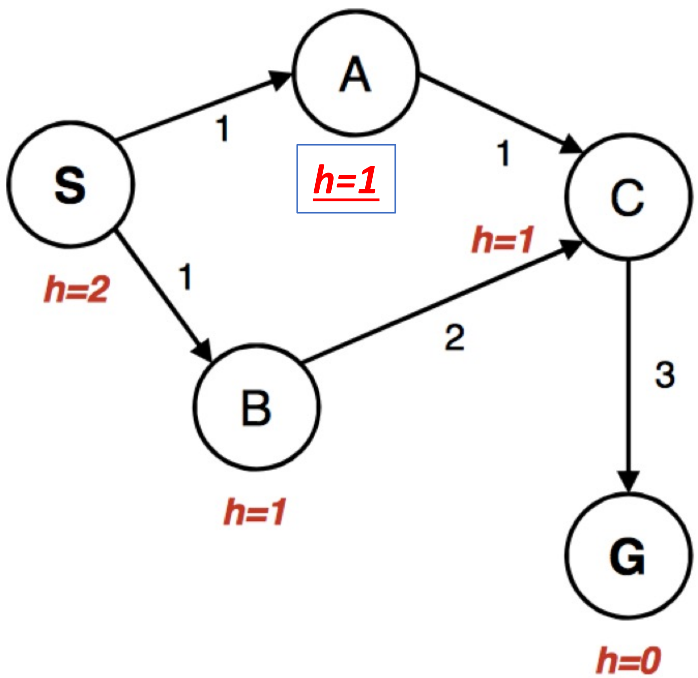
$$d(A) - d(C) = 1$$

but

$$h(A) - h(C) = 3$$

To fix this, we need to reduce $h(A)$ so that $h(A) - h(C) \leq 1$.

A* with a consistent heuristic



Consistent because:

$$d(A) - d(C) = 1$$

and

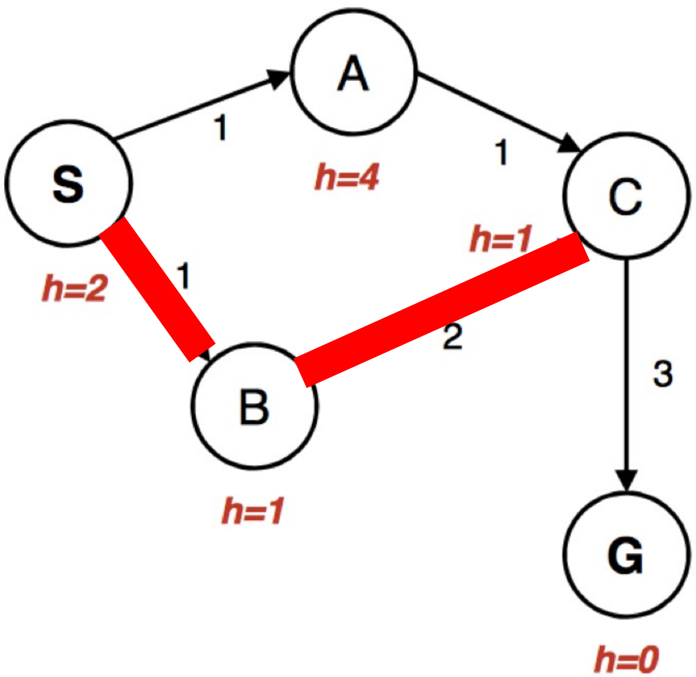
$$h(A) - h(C) = 0$$

Similarly

$$h(n) - h(p) \leq d(n) - d(p)$$

...for every pair of nodes n and p such that $d(n) \geq d(p)$.

A* with an inconsistent heuristic



Explored Set

S, B

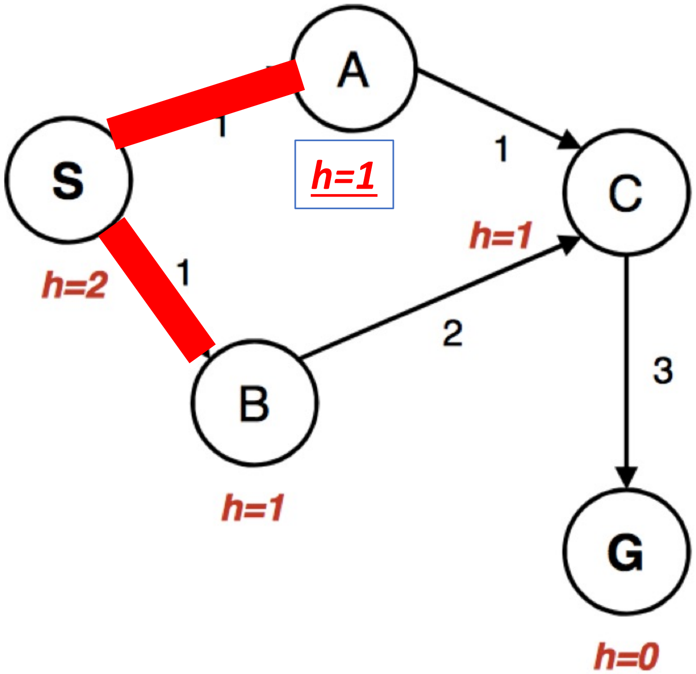
Frontier

A: g(A)+h(A)=5, g(A)=1, parent=S

C: g(C)+h(C)=4, g(C)=3, parent=B

Expand: C

A* with a consistent heuristic



Explored Set

S, B

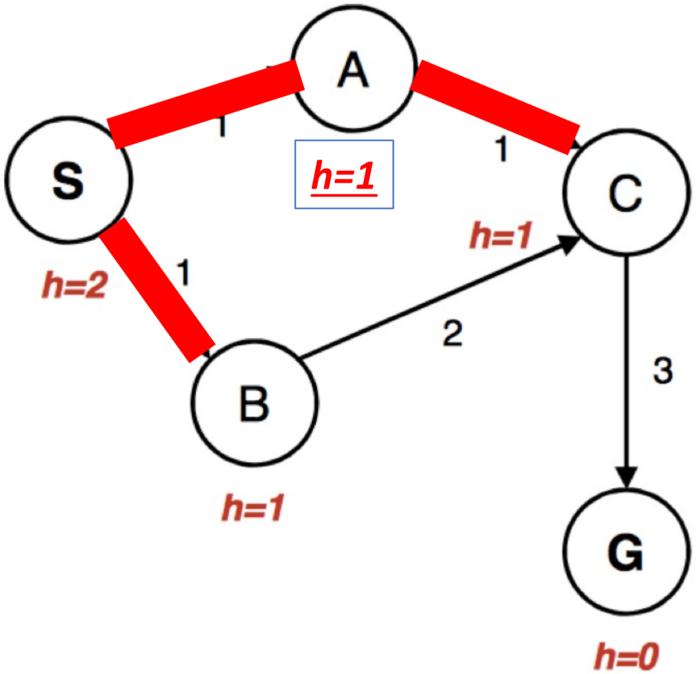
Frontier

A: g(A)+h(A)=**2**, g(A)=1, parent=S

C: g(C)+h(C)=4, g(C)=3, parent=B

Expand: **A**

A* with a **consistent** heuristic



Explored Set

S, B, **A**

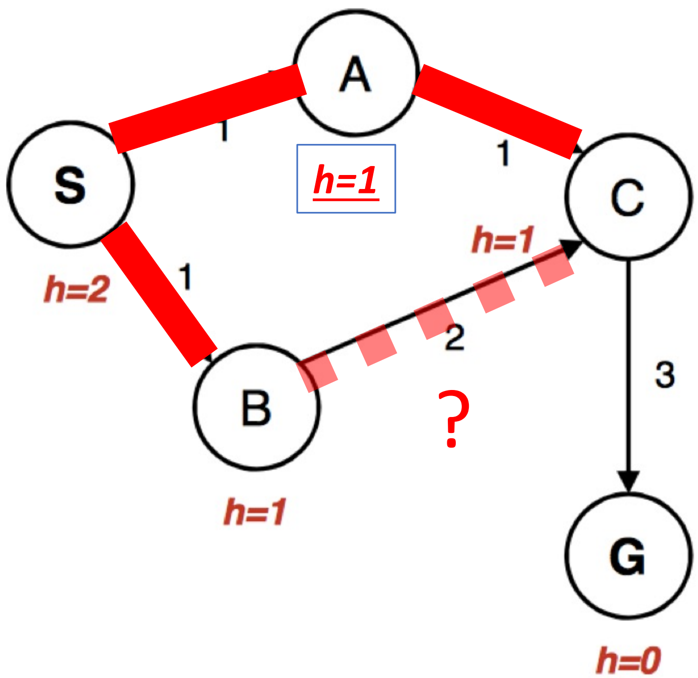
Frontier

C: g(C)+h(C)=4, g(C)=3, parent=B

C: g(C)+h(C)=**3**, g(C)=2, parent=**A**

Expand: the copy of **C** that has A as its parent.

A* with a **consistent** heuristic



Explored Set

S, B, **A,** C

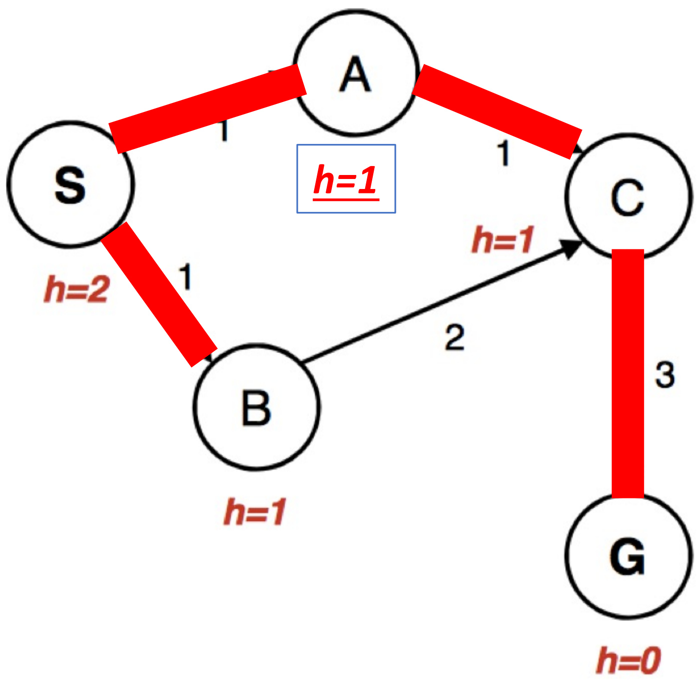
Frontier

C: g(C)+h(C)=4, g(C)=3, parent=B

G: g(G)+h(G)=**5**, parent=C

Expand: The copy of C that has B as its parent. But C is already in the explored set, and since our heuristic is consistent, we know that the new path (with B as parent) has a higher cost than the old path (with A as parent), so we can safely ignore the new path.

A* with a consistent heuristic



Explored Set

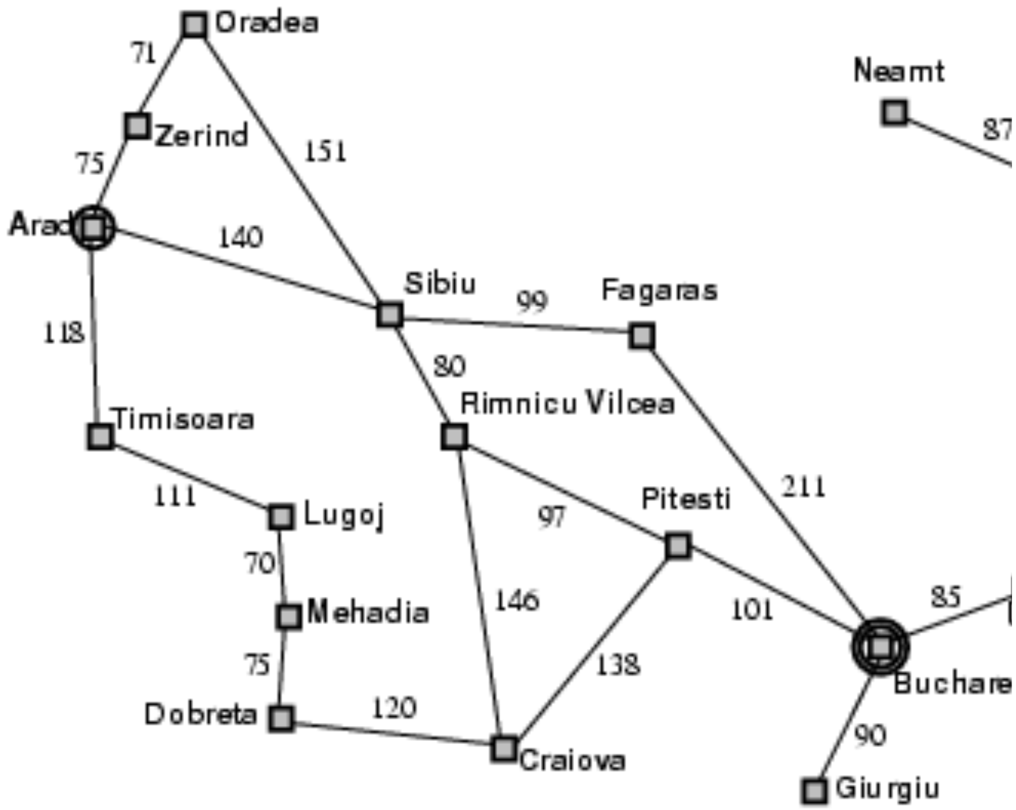
S, B, **A,** C

Frontier

G: g(G)+h(G)=**5**, parent=C

Expand: G

Admissible heuristic example: Romania



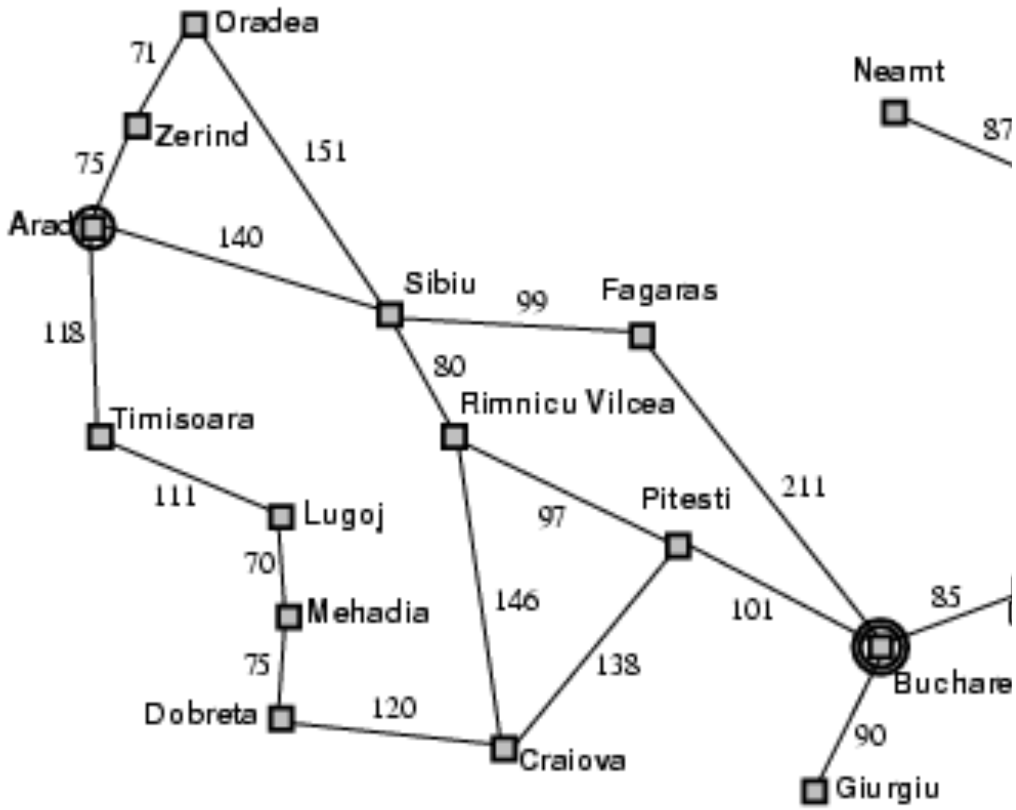
Admissible:

$$h(n) \leq d(n)$$

Example:

$$d(\text{Sibiu}) \ = \ 278$$
$$h(\text{Sibiu}) \ \leq \ 278$$

Consistent heuristic example: Romania



Consistent:

$$h(n) - h(p) \leq d(n) - d(p)$$

Example:

$$d(\text{Arad}) - d(\text{Sibiu}) = 140$$
$$h(\text{Arad}) - h(\text{Sibiu}) \leq 140$$

The heuristic difference is always less than or equal to the cost of the action.

Can you use this in the MP?

- Maybe.
- In the MP, every action has a cost of exactly 1!
- …so a consistent heuristic would be one such that, for every pair of neighboring states n and p, $h(n) - h(p) \leq 1$.
- Manhattan distance satisfies this condition.
- There are good heuristics for parts 3 and 4 that don't satisfy this condition. If your heuristic is not consistent, just make sure that you use an explored dict, instead of an explored set.

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1. Admissible heuristics
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The trivial case: h(n)=0

- A heuristic is **admissible** if and only if $d(n) \geq h(n)$ for every n.
- A heuristic is **consistent** if and only if $d(n, p) \geq h(n) - h(p)$ for every n and p.

- Both criteria are satisfied by $h(n) = 0$.

UCS = A* with h(n)=0

- Suppose we choose $h(n) = 0$
- Then the frontier is a priority queue sorted by
$$g(n) + h(n) = g(n)$$
- In other words, the first node we pull from the queue is the one that's closest to START!! (The one with minimum $g(n)$).
- **Uniform Cost Search** is **A* Search** with the heuristic $h(n) = 0$ for all states.

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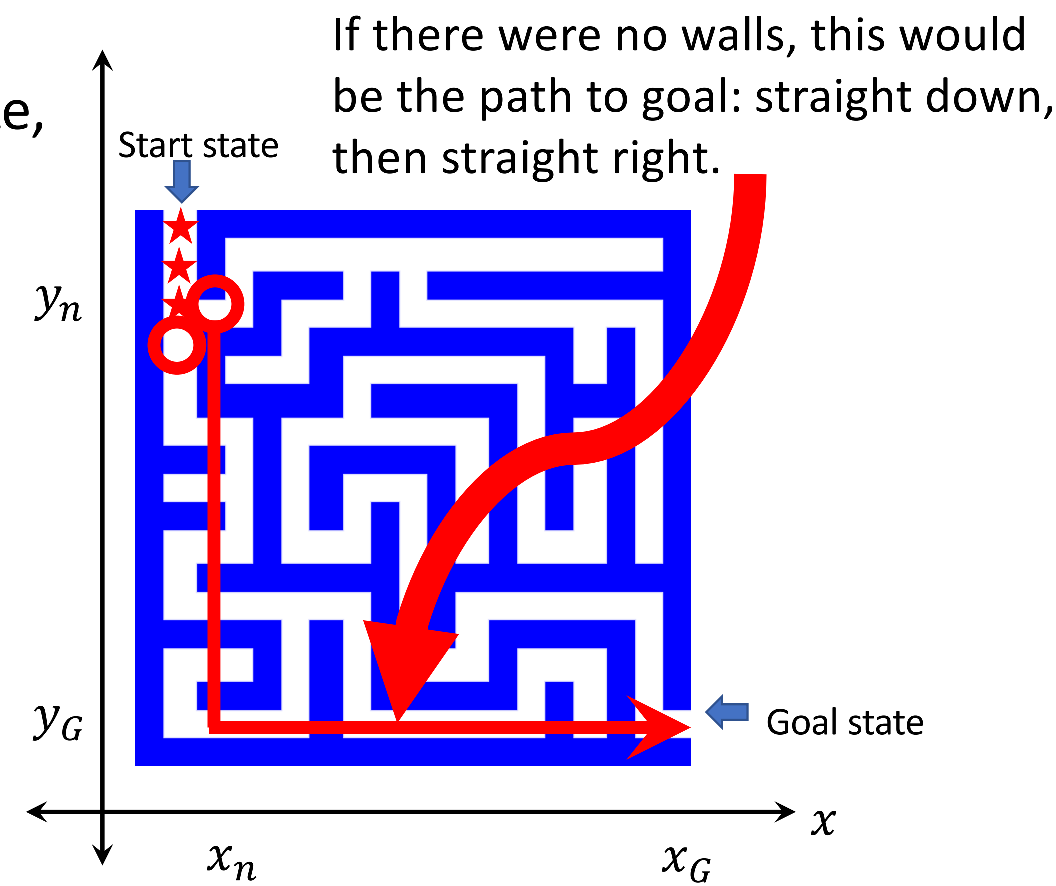
Heuristics from relaxed problems

- A problem with fewer restrictions on the actions is called a relaxed problem
- In most problems, having fewer restrictions on your action means that you can reach the goal faster.
- So designing a heuristic is usually the same as finding a relaxed problem that makes it easy to calculate the distance to goal.

Relaxed heuristic example: Manhattan distance

If there were no walls in the maze, then the number of steps from position (x_n, y_n) to the goal position (x_G, y_G) would be

$$h(n) = |x_n - x_G| + |y_n - y_G|$$

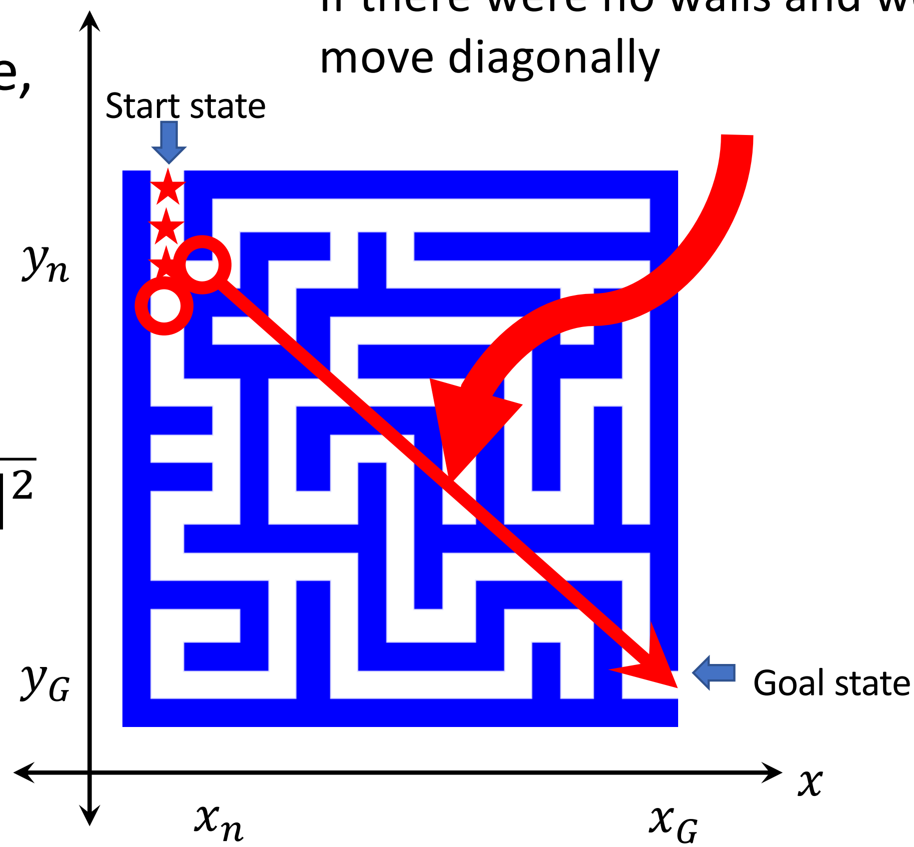


Relaxed heuristic example: Euclidean distance

If there were no walls in the maze, and we could move diagonally, then the number of steps from position (x_n, y_n) to the goal position (x_G, y_G) would be

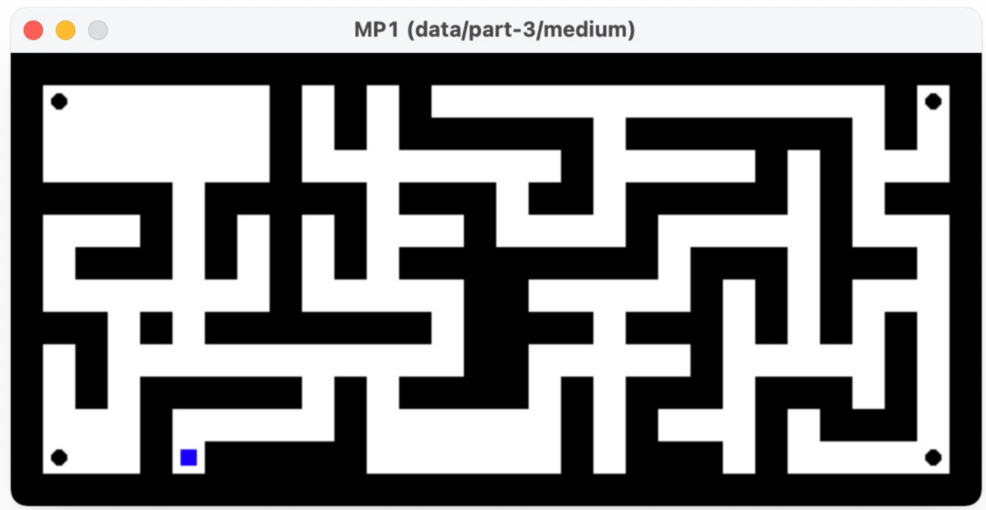
$$h(n) = \sqrt{|x_n - x_G|^2 + |y_n - y_G|^2}$$

If there were no walls and we could move diagonally



Relaxed heuristic example: Corner dots

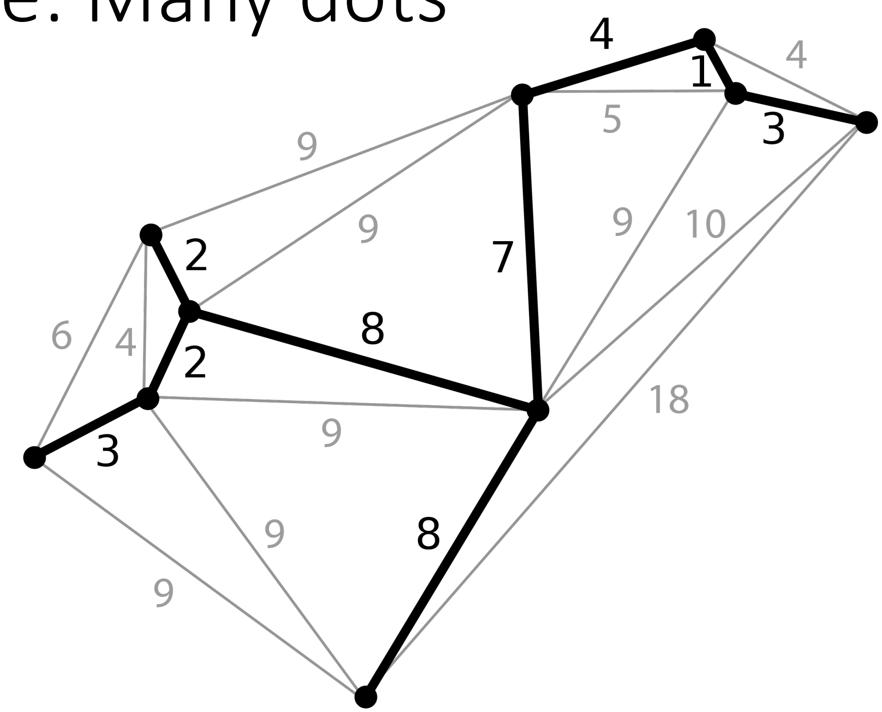
Suppose that, instead of touching ALL of the waypoints, you only had to touch the most extreme waypoints?



Relaxed heuristic example: Many dots

Suppose that, after you reached a waypoint, you could magically fly back to the nearest branch in the minimum spanning tree?

In other words, you only have to go one-way from where you are to the waypoint – you don't have to come back again.



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Which heuristic is better

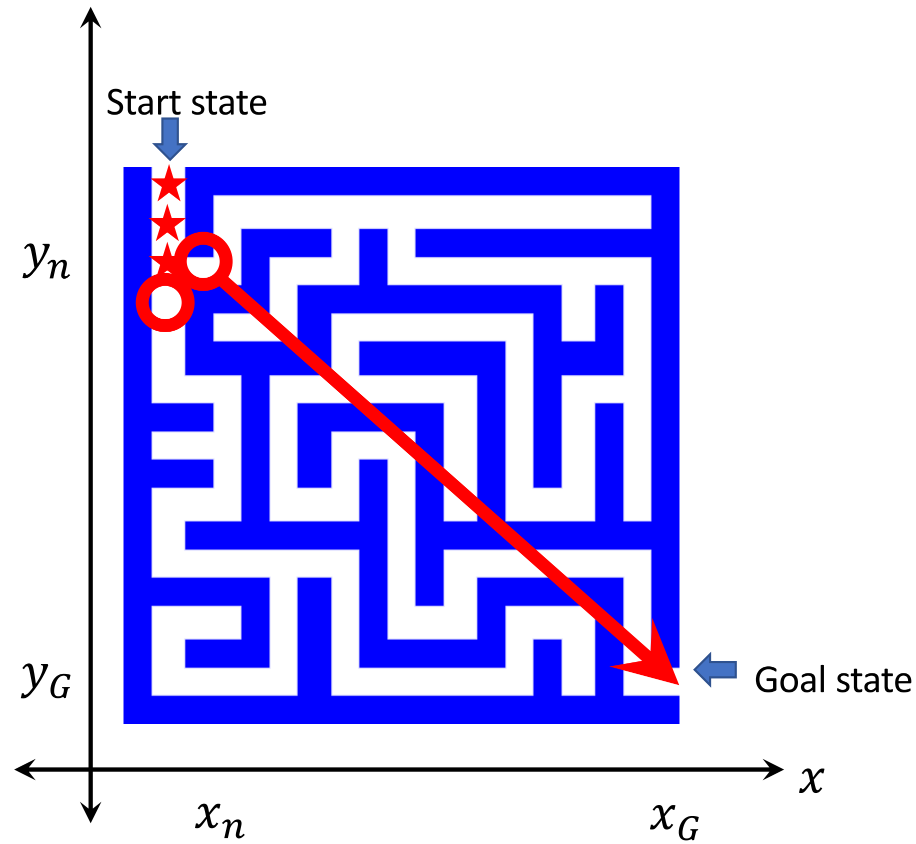
- If Euclidean distance and Manhattan distance are both admissible heuristics for the single-waypoint maze problem, which one is better?

- Computational complexity of A*: If $c(G)$ is true cost of the best path to goal, then A* evaluates every n for which $g(n) + h(n) \leq c(G)$

- How to minimize computational complexity: make $h(n)$ as large as possible, subject to the constraint that $h(n) \leq d(n)$.

Euclidean distance

$$h_2(n) = \sqrt{|x_n - x_G|^2 + |y_n - y_G|^2}$$

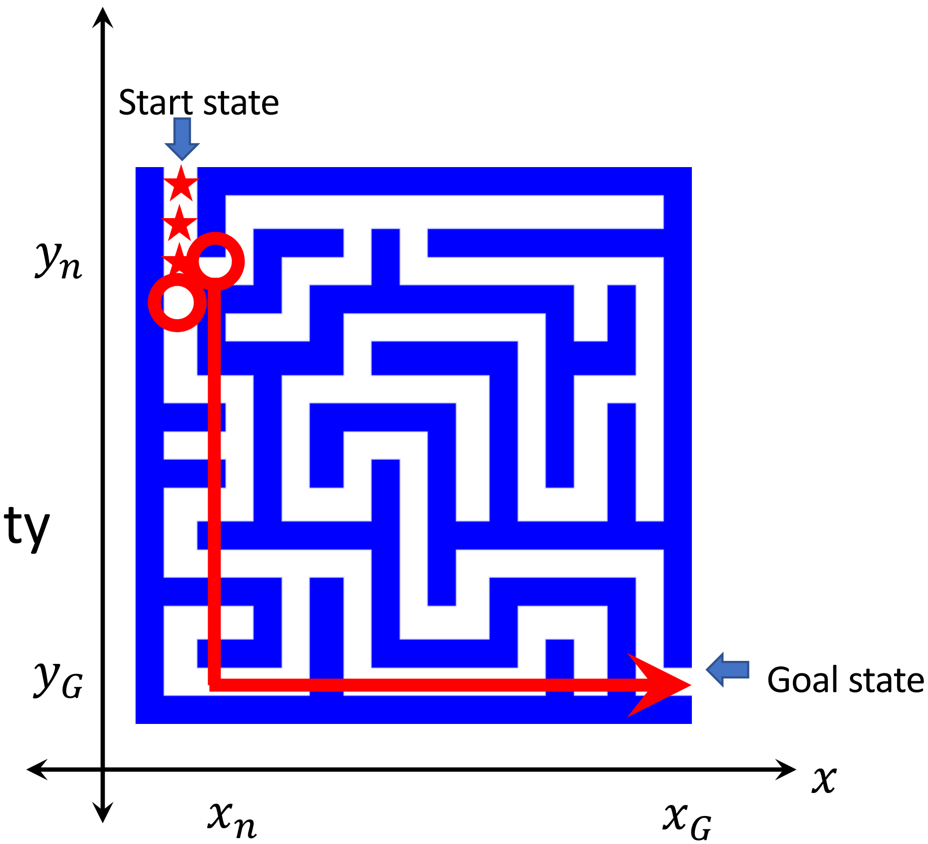


Manhattan distance

$$h_1(n) = |x_n - x_G| + |y_n - y_G|$$

$$h_1(n) \geq h_2(n)$$

Using $h_1(n)$, there will be fewer nodes with $g(n) + h(n) \leq c(G)$. Therefore, computational complexity is lower. Therefore $h_1(n)$ is better.

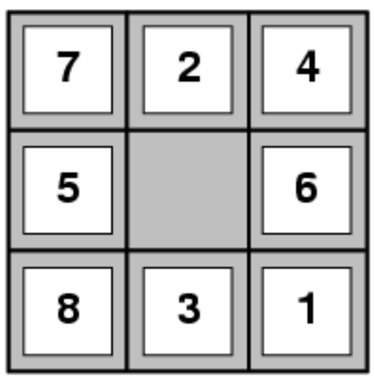


Dominance

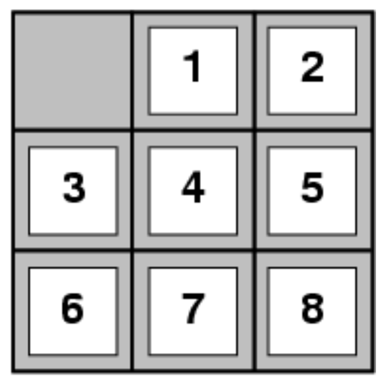
- If $h_2(n) \geq h_1(n)$ for all n, (both admissible) then h_2 **dominates** h_1
- As long as they're both admissible, they will both find the optimum path.
- But $h_2(n)$ will require less computation to find it.

Example: the 8-puzzle

- Problem statement: given a shuffled set of numbers (left), re-arrange them in order (right).
- State: ordering of the numbers and of the space.
- Possible actions: swap the space with any of its neighbors.
- Like traveling salesman, this is an NP-complete problem.



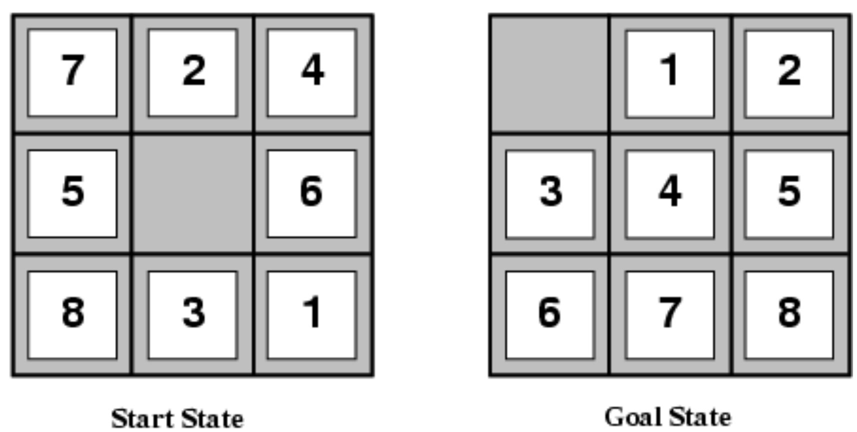
Start State



Goal State

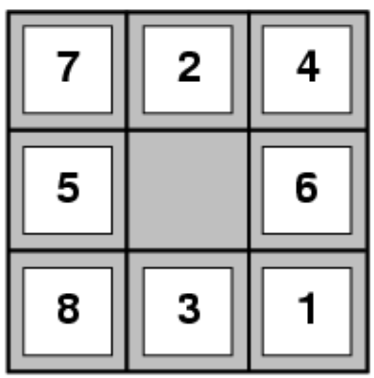
8-puzzle: Heuristic $h_1(n)$

- Suppose that, on each step, we could move any tile, anywhere on the board, regardless of where other tiles were.
- Then $h_1(n) =$ # tiles that need to be moved.
- Example below: $h_1(n) = 8$

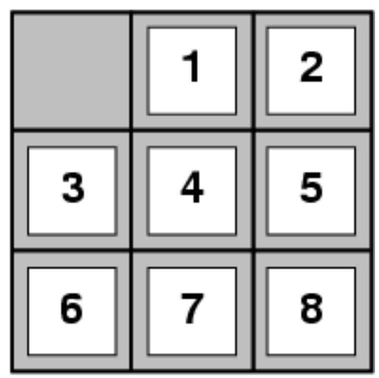


8-puzzle: Heuristic $h_2(n)$

- Suppose that, on each step, we could move any tile by just one step horizontally or vertically, regardless of whether there are other tiles in the way.
- Then $h_2(n) \; = $ sum of Manhattan distances from the current positions of each tile to their target positions (notice: $h_2(n) \geq h_1(n)$)
- Example below: $h_2(n) = 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2 = 18$



Start State



Goal State

Dominance

Experiment results reported by Svetlana Lazebnik

Typical search costs for the 8-puzzle (average number of nodes expanded for different solution depths):

- d=12 BFS expands 3,644,035 nodes
  $A^*(h_1)$ expands 227 nodes
  $A^*(h_2)$ expands 73 nodes

- d=24 BFS expands 54,000,000,000 nodes
  $A^*(h_1)$ expands 39,135 nodes
  $A^*(h_2)$ expands 1,641 nodes

Combining heuristics

- Suppose we have a collection of admissible heuristics $h_1(n)$, $h_2(n)$, …, $h_m(n)$, but none of them dominates the others
- How can we combine them?

$$h(n) = \max\{h_1(n), h_2(n), \ldots, h_m(n)\}$$

Outline of lecture

1. Admissible heuristics: $h(n) \leq d(n)$
2. Consistent heuristics: $h(n) - h(p) \leq d(n) - d(p)$
3. The zero heuristic: Uniform Cost Search: $h(n) = 0$
4. Relaxed heuristics: $h(n)$ is the $d(n)$ from a problem with fewer rules.
5. Dominant heuristics: if $h_2(n) \leq h_1(n)$ and both are admissible, then $h_1(n)$ has lower computational complexity

Five search strategies

Algorithm	Complete?	Optimal?	Time complexity	Space complexity	Implement the Frontier as a…
BFS	Yes	If all step costs equal	$O\{b^d\}$	$O\{b^d\}$	Queue
DFS	No	No	$O\{b^m\}$	$O\{bm\}$	Stack
UCS	Yes	Yes	#nodes s.t. $g(n) \leq c(G)$	#nodes s.t. $g(n) \leq c(G)$	Priority Queue: $g(n)$
Greedy	No	No	$O\{b^m\}$	$O\{b^m\}$	Priority Queue: $h(n)$
A*	Yes	Yes	#nodes s.t. $g(n) + h(n) \leq c(G)$	#nodes s.t. $g(n) + h(n) \leq c(G)$	Priority Queue: $h(n) + g(n)$