# CS440/ECE448 Lecture 29: Markov Decision Processes

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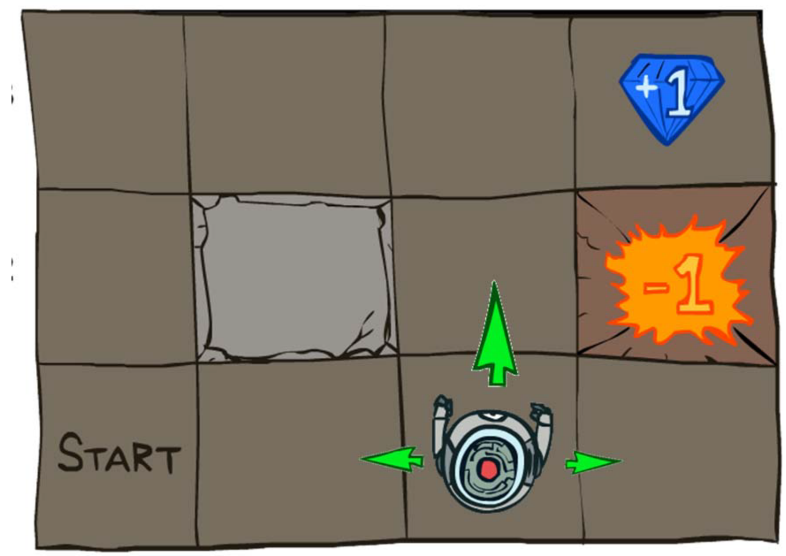
Including slides by Svetlana Lazebnik, 11/2016

Including many figures by Peter Abbeel and Dan Klein, UC Berkeley CS 188




Grid World

Invented and drawn by Peter Abbeel and Dan Klein, UC Berkeley CS 188

# Markov Decision Processes

- In HMMs, we see a sequence of observations and try to reason about the underlying state sequence
  - There are no actions involved
- But what if we have to take an action at each step that, in turn, will affect the state of the world?

# Markov Decision Processes

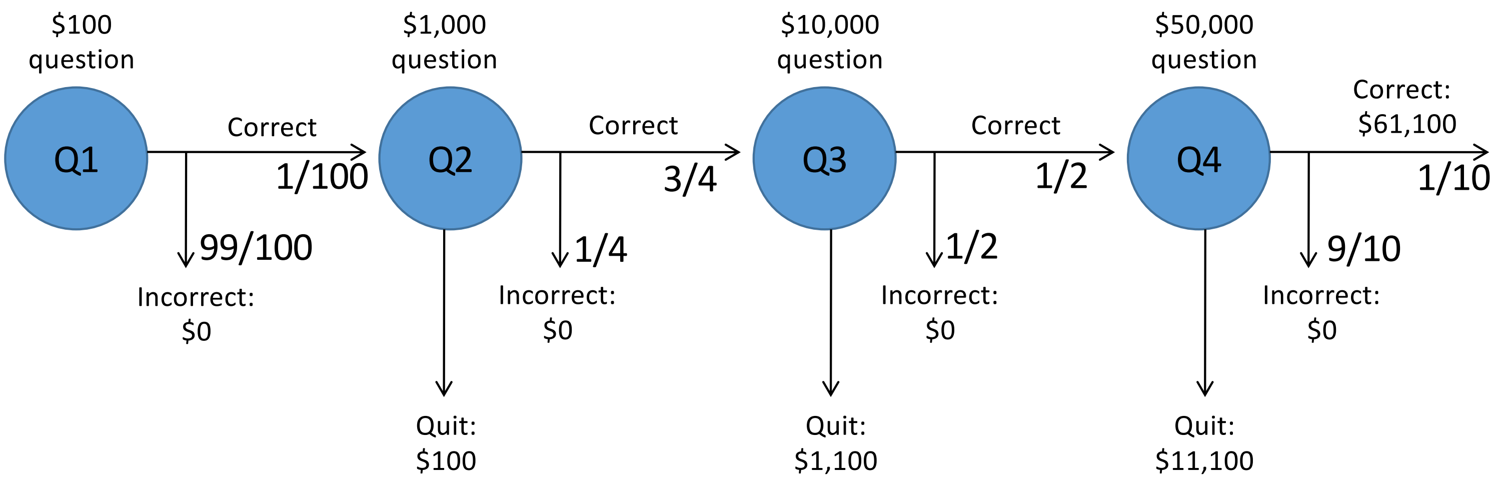
- Components that define the MDP. Depending on the problem statement, you either know these, or you learn them from data:
  - **States** $s$, beginning with initial state $s_0$
  - **Actions** $a$
    - Each state $s$ has actions $A(s)$ available from it
  - **Transition model** $P(s' \mid s, a)$
    - *Markov assumption*: the probability of going to $s'$ from $s$ depends only on $s$ and $a$ and not on any other past actions or states
  - **Reward function** $R(s)$
- **Policy – the "solution" to the MDP:**
  - $\pi(s) \in A(s)$: the action that an agent takes in any given state

# Overview

- First, we will look at how to “solve” MDPs, or find the optimal policy when the transition model and the reward function are known
- Second, we will consider **reinforcement learning**, where we don’t know the rules of the environment or the consequences of our actions

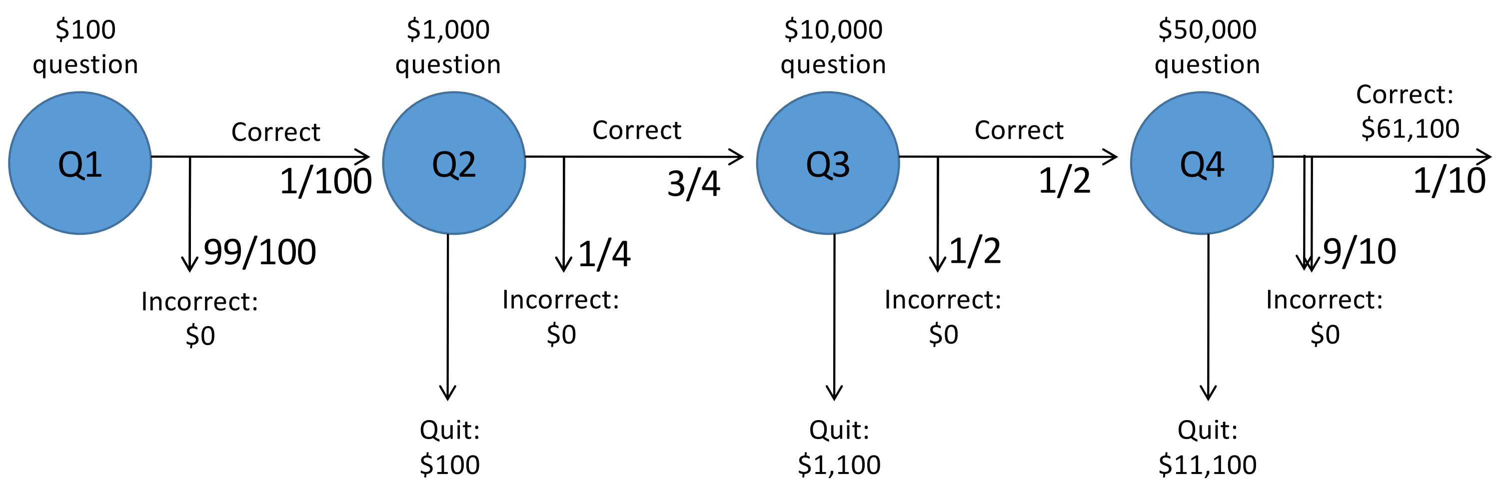
# Game show

- A series of questions with increasing level of difficulty and increasing payoff
- Decision: at each step, take your earnings and quit, or go for the next question
  - If you answer wrong, you lose everything



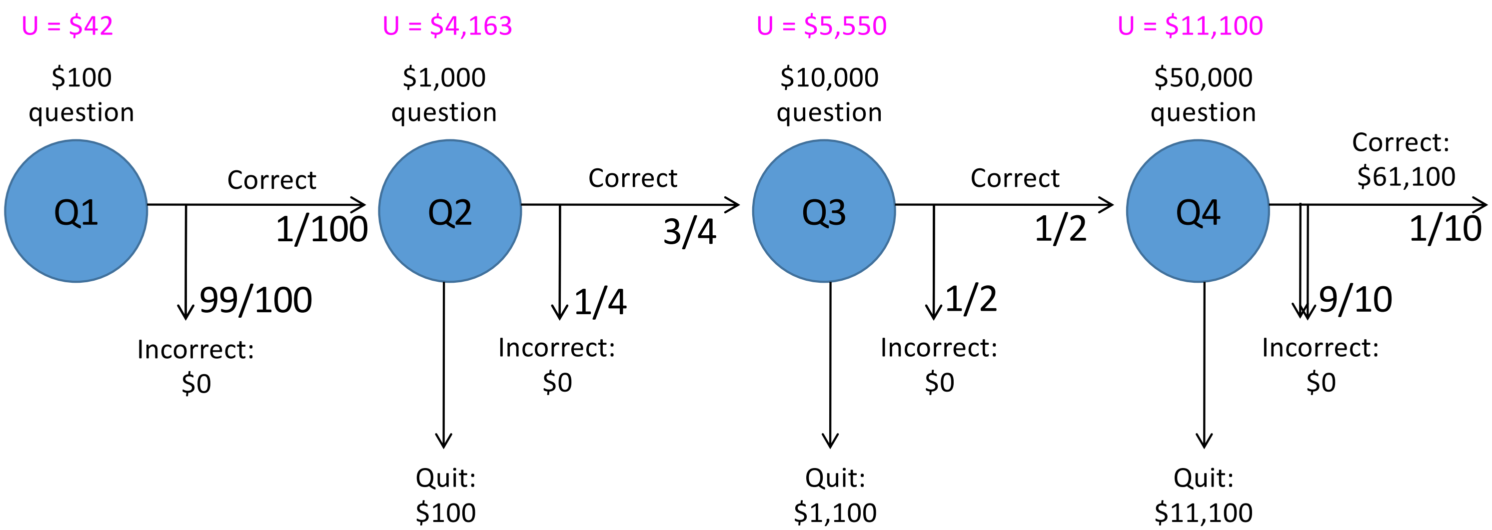
# Game show

- Consider $50,000 question
  - Probability of guessing correctly: 1/10
  - Quit or go for the question?
- What is the expected payoff for continuing?
  0.1 * 61,100 + 0.9 * 0 = 6,110
- What is the optimal decision?

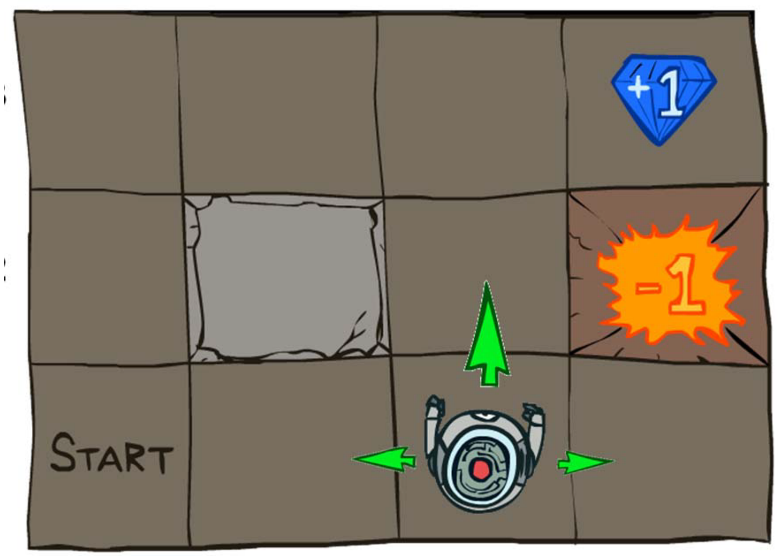


# Game show

- What should we do in Q3?
  - Payoff for quitting: $1,100
  - Payoff for continuing: 0.5 * $11,100 = $5,550
- What about Q2?
  - $100 for quitting vs. $4,162 for continuing
- What about Q1?

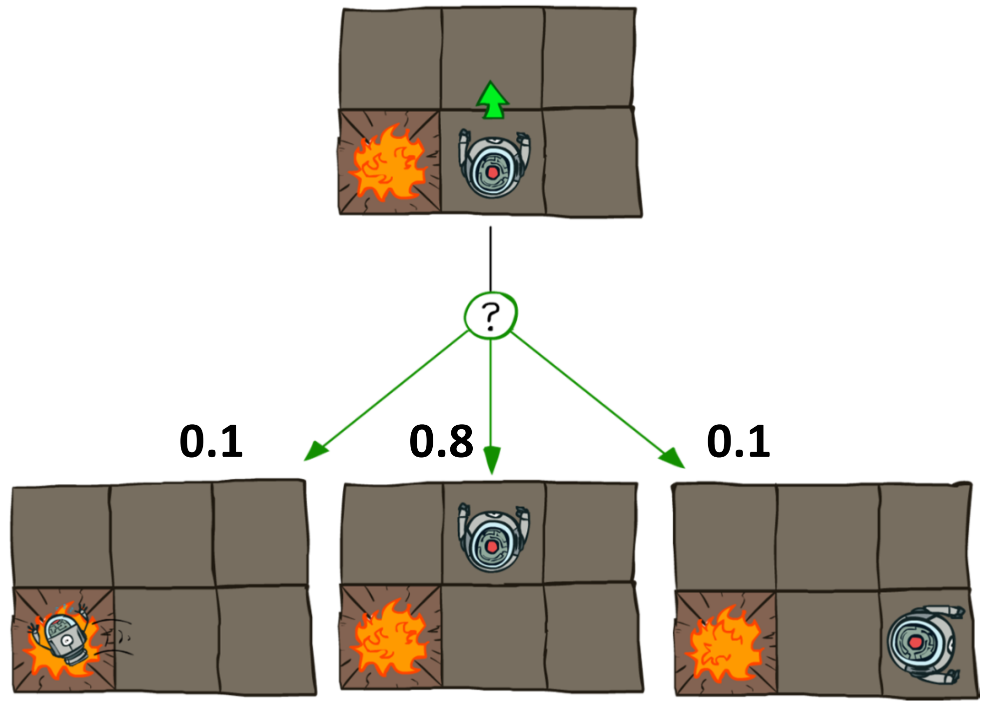


# Grid world



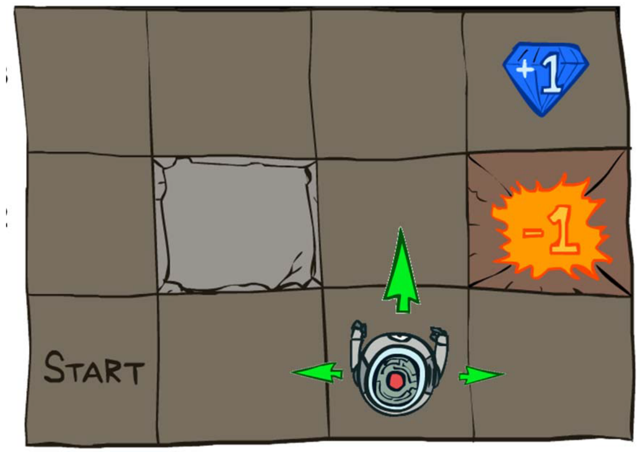

R(s) = -0.04 for every non-terminal state

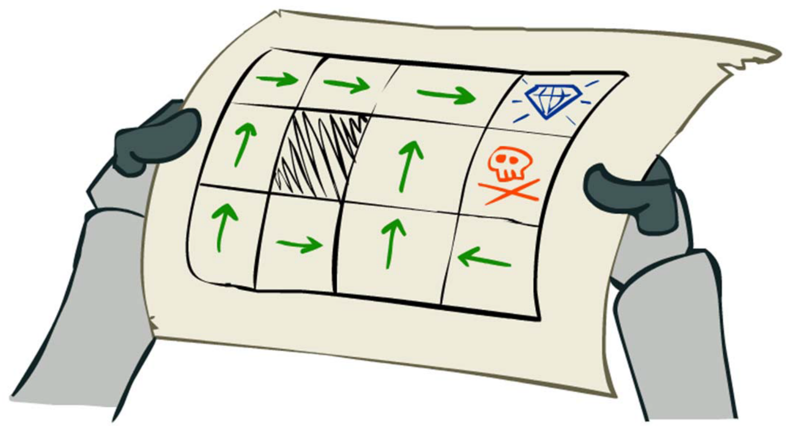
Transition model:




Source: P. Abbeel and D. Klein

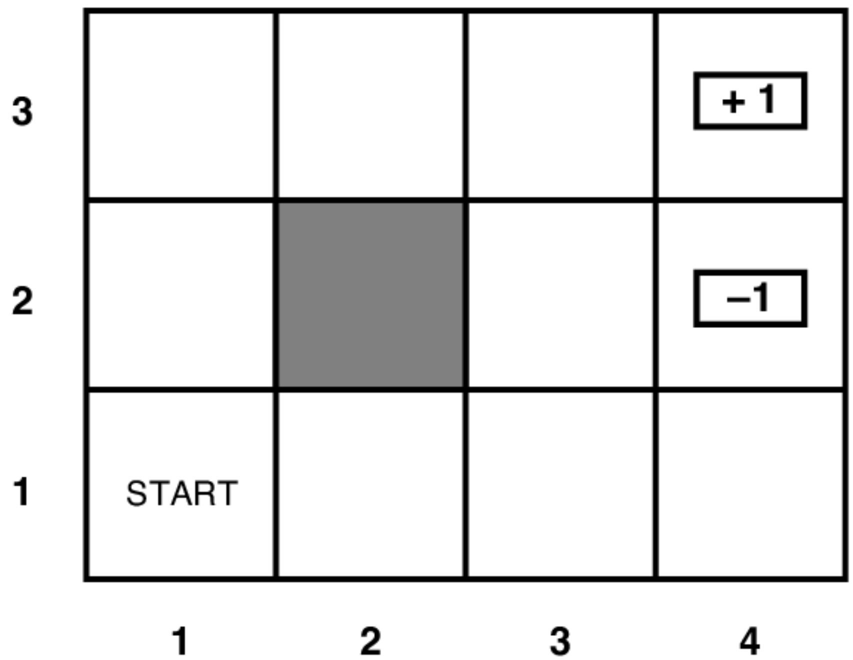
# Goal: Policy



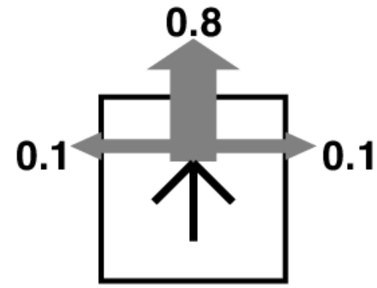



Source: P. Abbeel and D. Klein

# Grid world

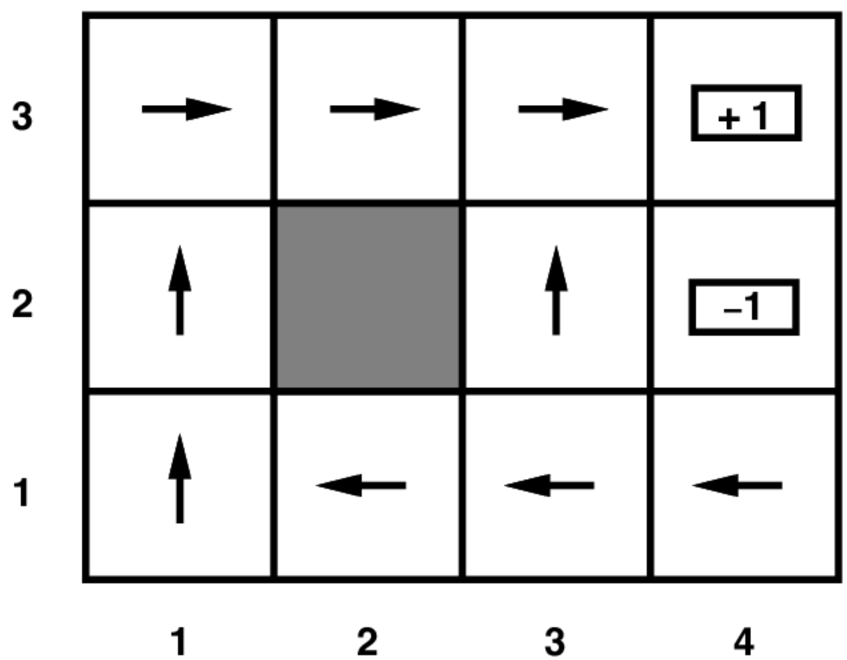



Transition model:




R(s) = -0.04 for every non-terminal state

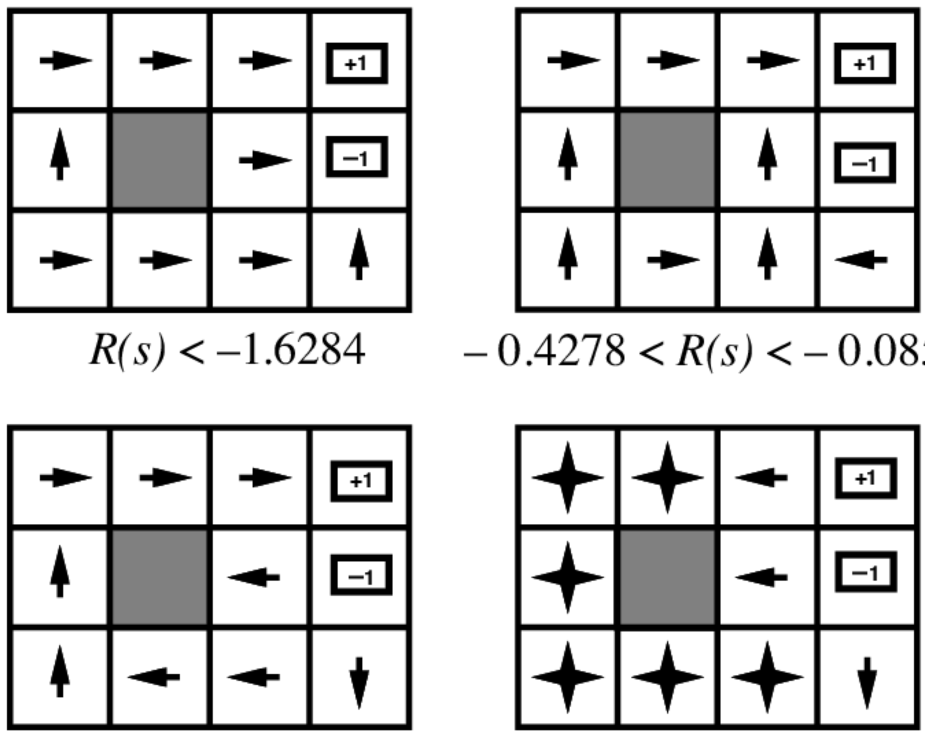
# Grid world




Optimal policy when R($s$) = -0.04 for every non-terminal state

# Grid world

- Optimal policies for other values of R(s):




$R(s) < -1.6284$

$-0.4278 < R(s) < -0.0850$

$-0.0221 < R(s) < 0$

$R(s) > 0$

# Solving MDPs

- MDP components:
  - **States** s
  - **Actions** a
  - **Transition model** P(s' | s, a)
  - **Reward function** R(s)
- The solution:
  - **Policy** $\pi$(s): mapping from states to actions
  - How to find the optimal policy?

# Maximizing expected utility

- The optimal policy $\pi(s)$ should maximize the *expected utility* over all possible state sequences produced by following that policy:

$$\sum_{\substack{state\ sequences\\ starting\ from\ s_0}} P\big(sequence | s_0, a = \pi(s_0)\big) U(sequence)$$

- How to define the utility of a state sequence?
  - Sum of rewards of individual states
  - Problem: infinite state sequences

# Utilities of state sequences

- Normally, we would define the utility of a state sequence as the sum of the rewards of the individual states
- **Problem:** infinite state sequences
- **Solution:** *discount* the individual state rewards by a factor $\gamma$ between 0 and 1:

$$U([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$$

$$= \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma} \qquad (0 < \gamma < 1)$$

  - Sooner rewards count more than later rewards
  - Makes sure the total utility stays bounded
  - Helps algorithms converge

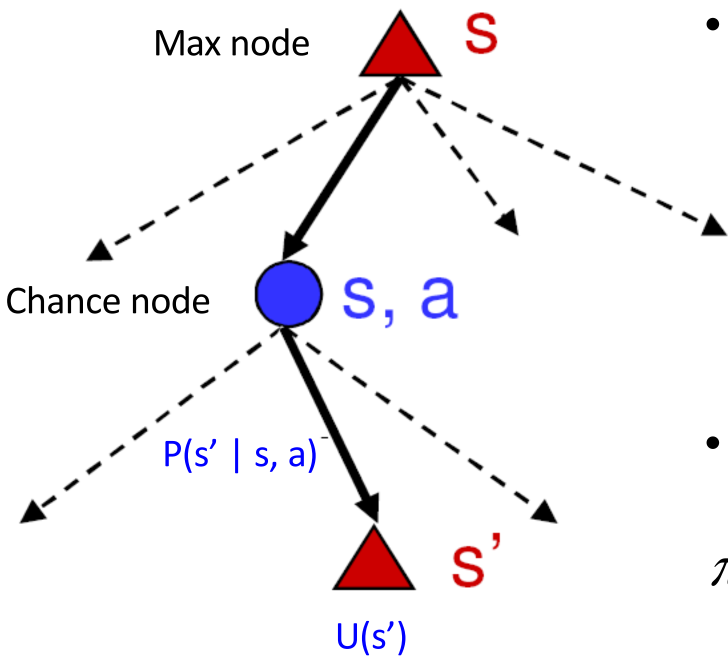
# Utilities of states

- Expected utility obtained by policy π starting in state s:

$$U^{\pi}(s) = \sum_{\substack{state\ sequences \\ starting\ from\ s}} P\big(sequence \,|\, s, a = \pi(s)\big) U(sequence)$$

- The "true" utility of a state, denoted U(s), is the *best possible* expected sum of discounted rewards
  - if the agent executes the *best possible* policy starting in state s
- Reminiscent of minimax values of states…

# Finding the utilities of states




- If state s' has utility U(s'), then what is the expected utility of taking action **a** in state **s**?

$$\sum_{s'} P(s' | s, a) U(s')$$

- How do we choose the optimal action?

$$\pi^*(s) = \arg\max_{a \in A(s)} \sum_{s'} P(s' | s, a) U(s')$$

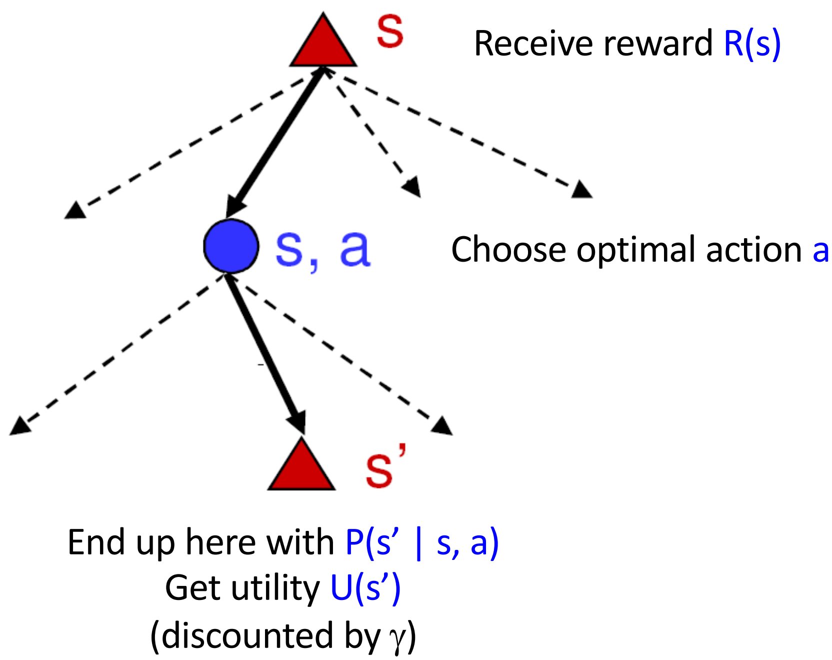
- What is the recursive expression for **U(s)** in terms of the utilities of its successor states?

$$U(s) = R(s) + \gamma \max_a \sum_{s'} P(s' | s, a) U(s')$$

# The Bellman equation

- Recursive relationship between the utilities of successive states:

$$\boxed{U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')}$$



# The Bellman equation

- Recursive relationship between the utilities of successive states:

$$\boxed{U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')}$$

- For *N* states, we get *N* **nonlinear** equations in *N* unknowns
  - Known quantities: $P(s'|s, a)$, $R(s)$, and $\gamma$. Unknowns: $U(s)$.
  - Solving these N equations solves the MDP.
  - Nonlinear -> no closed-form solution.
    - If it weren't for the "max," this would be N linear equations in N unknowns. We could solve it by just inverting an NxN matrix.
    - The "max" means that there is no closed-form solution. Need to use an iterative solution method, which might not converge to the globally optimum soluton.
  - Two solution methods: **value iteration** and **policy iteration**

# Method 1: Value iteration

- Start out with iteration $i = 0$, every $U_i\ (s)\ =\ 0$
- Iterate until convergence
  - During the $i^{\text{th}}$ iteration, update the utility of each state according to this rule:

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U_i(s')$$

- In the limit of infinitely many iterations, guaranteed to find the correct utility values.
  - Error decreases exponentially, so in practice, don't need an infinite number of iterations…

# Value Iteration: Iteration 1

$$U_1(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) U_0(s')$$

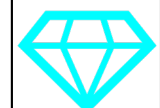
$U_1(s)$

| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $-0.04$ |  |
| $-0.04$ | | $-0.04$ |  |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.04$ |

$R(s)$

| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $-0.04$ |  |
| $-0.04$ | | $-0.04$ |  |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.04$ |

$U_0(s)$

| | | | |
|---|---|---|---|
| 0 | 0 | 0 |  |
| 0 | | 0 |  |
| 0 | 0 | 0 | 0 |

# Value Iteration: Iteration 2

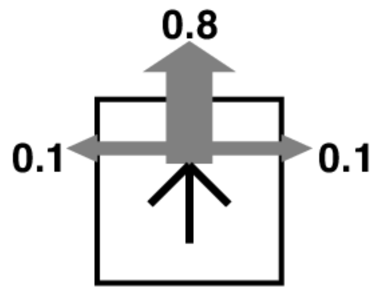
$$U_1(s)$$

| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $-0.04$ |  |
| $-0.04$ | | $-0.04$ |  |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.04$ |

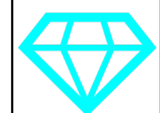
# Value Iteration: Iteration 2

$$U_2(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) U_1(s')$$

Transition model:




$U_1(s)$

| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $-0.04$ |  |
| $-0.04$ | | $-0.04$ |  |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.04$ |

$$P(s'|s,\text{up}) = \begin{cases} 0.8 & s' = \text{up from } s \text{ (if no wall)} \\ 0.1 & s' = \text{left from } s \text{ (if no wall)} \\ 0.1 & s' = \text{right from } s \text{ (if no wall)} \end{cases}$$

# Value Iteration: Iteration 2

$$U_2(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a)U_1(s')$$

$U_2(s)\ (\gamma = 1)$

| | | | |
|---|---|---|---|
| $-0.08$ | $-0.08$ | $+0.75$ | |
| $-0.08$ | | $-0.08$ | |
| $-0.08$ | $-0.08$ | $-0.08$ | $-0.08$ |

$\sum_{s'} P(s'|s,\text{down})U_1(s)$

| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $+0.06$ | |
| $-0.04$ | | $-0.14$ | |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.04$ |

$\sum_{s'} P(s'|s,\text{up})U_1(s)$

| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $+0.06$ | |
| $-0.04$ | | $-0.14$ | |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.81$ |

$\sum_{s'} P(s'|s,\text{left})U_1(s)$

| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $-0.04$ | |
| $-0.04$ | | $-0.04$ | |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.14$ |

$\sum_{s'} P(s'|s,\text{right})U_1(s)$

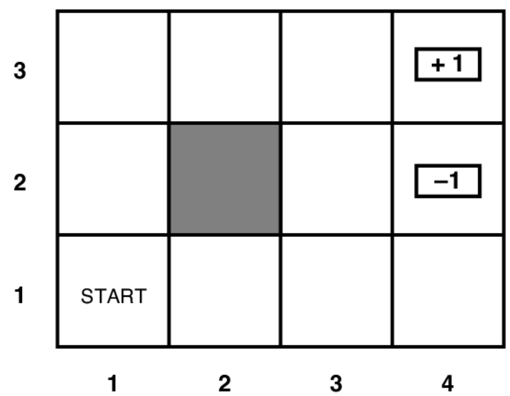
| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $+0.79$ | |
| $-0.04$ | | $-0.81$ | |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.14$ |

$U_1(s)$

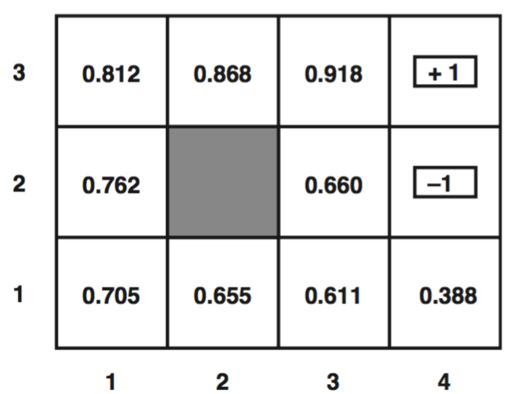
| | | | |
|---|---|---|---|
| $-0.04$ | $-0.04$ | $-0.04$ | |
| $-0.04$ | | $-0.04$ | |
| $-0.04$ | $-0.04$ | $-0.04$ | $-0.04$ |

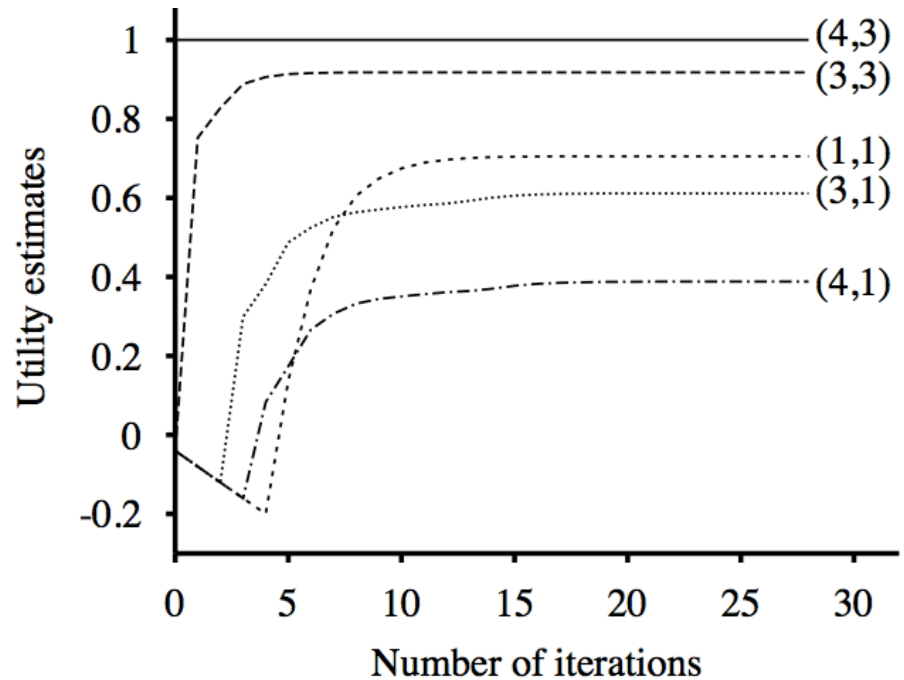
# Value iteration

Input (non-terminal R=-0.04)

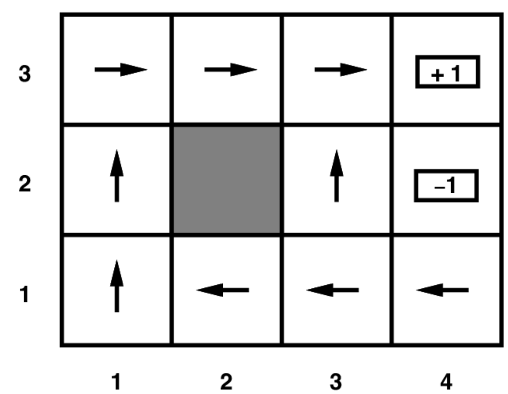



Utilities with discount factor 1




Final policy



# Method 2: Policy Iteration

- Start with some initial policy $\pi_0$ and alternate between the following steps:
  - **Policy Evaluation:** calculate the utility of every state under the assumption that the given policy is fixed and unchanging.
  - **Policy Improvement:** calculate a new policy $\pi_{i+1}$ based on the updated utilities.

- Notice it's kind of like gradient descent in neural networks:
  - Policy evaluation: Find ways in which the current policy is suboptimal
  - Policy improvement: Fix those problems

- Unlike Value Iteration, this is guaranteed to converge in a finite number of steps, as long as the state space and action set are both finite.

# Method 2: Policy Iteration

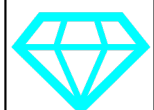
- **Policy Evaluation**: Given a fixed policy $\pi$, calculate the **policy-dependent utility**, $U^{\pi}(s)$, for every state $s$

$$\boxed{U^{\pi}(s) = R(s) + \gamma \sum_{s'} P(s' | s, \pi(s)) U^{\pi}(s')}$$

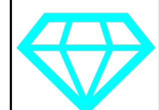
# Policy Iteration: Iteration 1

**Policy Evaluation:** $U^{\pi^0}(s) = R(s) + \gamma \sum_{s'} P(s'|s,a) U^{\pi^0}(s')$

$U^{\pi^0}(s)$

| | | | |
|---|---|---|---|
| $+0.50$ | $+0.69$ | $+0.74$ |  |
| $-0.65$ | | $-0.90$ |  |
| $-1.40$ | $-1.44$ | $-1.39$ | $-1.40$ |

$\pi^0(s)$

| | | | |
|---|---|---|---|
| $\rightarrow$ | $\rightarrow$ | $\rightarrow$ |  |
| $\rightarrow$ | | $\rightarrow$ |  |
| $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ |

# Why is Policy Evaluation easy, while Bellman Equation is hard?

- **Policy Evaluation**: Given a fixed policy π, calculate the **<u>policy-dependent utility</u>**, $U^\pi(s)$, for every state *s*

$$\boxed{U^\pi(s) = R(s) + \gamma \sum_{s'} P(s' | s, \pi(s)) U^\pi(s')}$$

- π(s) is fixed, therefore $P(s'|s,\pi(s))$ is an $N\times N$ matrix, therefore we can just invert the $N\times N$ matrix: $U^\pi(s) = \left(\mathrm{I} - \gamma P(s'|s,\pi(s))\right)^{-1} R(s)$
- Why is this "Policy Evaluation" formula so much easier to solve than the original Bellman equation?

$$\boxed{U(s) = R(s) + \gamma \max_{a\in A(s)} \sum_{s'} P(s' | s, a) U(s')}$$

# Method 2: Policy Iteration

- **Policy Evaluation**: Given a fixed policy π, calculate the **<u>policy-dependent utility</u>**, $U^{\pi}(s)$, for every state *s*

$$\boxed{U^{\pi}(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) U^{\pi}(s')}$$

- **Policy Improvement**: Given $U^{\pi}(s)$ for every state *s*, find an improved π(s)

$$\boxed{\pi^{i+1}(s) = \arg\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U^{\pi_i}(s')}$$

# Policy Iteration: Iteration 1

**Policy Evaluation:** $$U^{\pi^0}(s) = R(s) + \gamma \sum_{s'} P(s'|s,a) U^{\pi^0}(s')$$

**Policy Improvement:** $$\pi^1(s) = \underset{a}{\operatorname{argmax}} \sum_{s'} P(s'|s,a) U^{\pi^0}(s')$$

$\pi^1(s)$

| | | | |
|---|---|---|---|
| → | → | → | |
| ↑ | | ↑ | |
| ↑ | ↑ | ↑ | ↑ |

$U^{\pi^0}(s)$

| | | | |
|---|---|---|---|
| +0.50 | +0.69 | +0.74 | |
| −0.65 | | −0.90 | |
| −1.40 | −1.44 | −1.39 | −1.40 |

$\pi^0(s)$

| | | | |
|---|---|---|---|
| → | → | → | |
| → | | → | |
| → | → | → | → |

# Summary

- MDP defined by states, actions, transition model, reward function
- The "solution" to an MDP is the policy: what do you do when you're in any given state
- The Bellman equation tells the utility of any given state, and incidentally, also tells you the optimum policy.   The Bellman equation is N nonlinear equations in N unknowns (the policy), therefore it can't be solved in closed form.
- Value iteration:
  - At the beginning of the (i+1)'st iteration, each state's value is based on looking ahead i steps in time
  - … so finding the best action = optimize based on (i+1)-step lookahead
- Policy iteration:
  - Find the utilities that result from the current policy,
  - Improve the current policy