CS440/ECE448 Lecture 13: Random Variables

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Random Variables

- Expected Value
- Probability Mass Function
- Domain of a Random Variable
  - Domain Type: Categorical vs. Numerical
  - Domain Size: Finite vs. Countably Infinite vs. Uncountably Infinite
- Joint, Marginal, and Conditional Random Variables
  - Marginalization and Conditioning
  - Law of Total Probability
  - Random Vectors
  - Jointly Random Class and Measurement Variables
- Functions of Random Variables
  - Probability Mass Function
  - Expectation

Expected Value

Expected Value of a random variable = the average value, averaged over an infinite number of independent trials

Expected Value

Example: D = number of pips showing on a die

Expected Value of a random variable = the average value, averaged over an infinite number of independent trials

E[D] = lim($n \to \infty$) (1/n) * (1*(#times D=1)+2*(#times D=2)+…+6*(#times D=6))

= 1*(1/6) + 2*(1/6) + …. + 6*(1/6)

= (1+2+3+4+5+6)/6

= 3.5

Expected Value

Expected Value of a random variable = the average value, averaged over an infinite number of independent trials

= sum{ value * P(variable=value) }

Center of Mass (from physics)

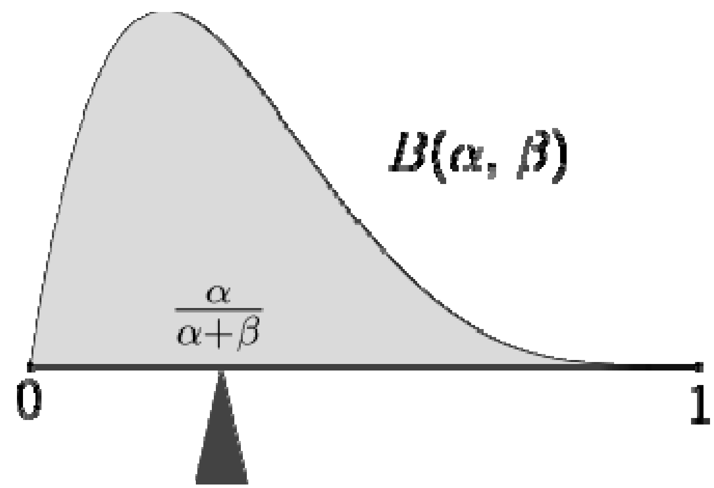
Center of Mass

= sum{ position * Mass(position) }

Expected Value = Center of Probability “Mass”

Expected Value of a random variable = the average value, averaged over an infinite number of independent trials

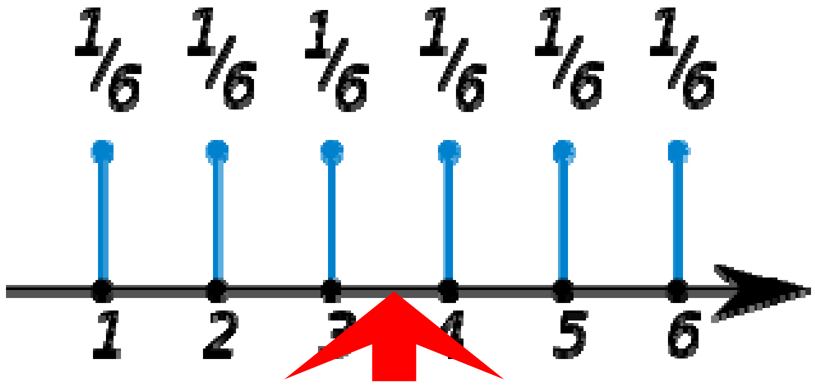
= sum{ value * P(variable=value) }



Wikipedia: “The mass of probability distribution is balanced at the expected value.”

Probability Mass Function

- The "Probability Mass Function" (pmf) of a random variable X is defined to be the function P(X=value), as a function of the different possible values.
- Why it's useful: expected value = center of mass.



Wikipedia: "The probability mass function of a fair die. All the numbers on the die have an equal chance of appearing on top when the die stops rolling." The **expected value** is 3.5.

Requirements for a Probability Mass Function

<u>Axioms of Probability</u>

P(A) ≥ 0 for every event A

1 = P(True)

$P(A \vee B) = P(A) + P(B) – P(A \wedge B)$

<u>Requirements for a PMF</u>

P(X=x) ≥ 0 for every x

1 = Sum_x P(X=x)

$P(X=x \vee X=y) = P(X=x)+P(X=y)$

Notice: the last one assumes that X=x and X=y are mutually exclusive events.

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Domain of a Random Variable

The "Domain" of a Random Variable is the set of its possible values.

Domain of a Random Variable

- The domain can be numerical. For example:
  - The number of pips showing on a die
  - The age, in years, of a person that you choose at random off the street
  - The number of days of sunshine in the month of March
  - The minimum temperature tonight, in degrees Celsius
- The domain can also be categorical. For example:
  - The color chosen by a spinner in the game of Twister
  - The color of the shirt worn by a person chosen at random
  - The type of weather tomorrow: { sunny, cloudy with no precipitation, raining, snowing, sleet }

Expectation and PMF

- Expected Value is only well defined for numerical domains.

E[X] = sum value * P(X=value)

- PMF is well defined even for categorical domains.

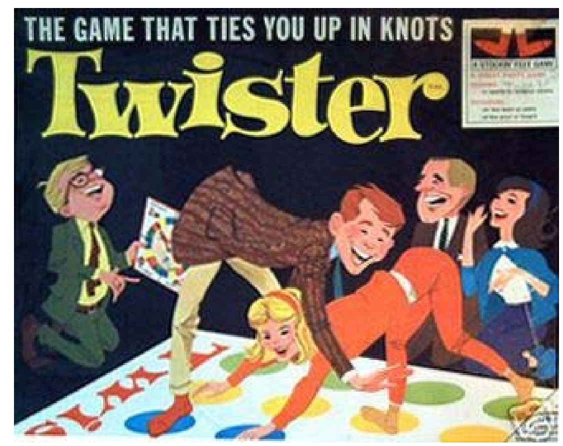
Example: X = color shown on the spinner

P(X=red) = (1/4)

P(X=blue) = (1/4)

P(X=green) = (1/4)

P(X=yellow) = (1/4)



Size of the Domain = # Different Possible Values

- Domain of a random variable can be finite.

Example: D = value, in dollars, of the next coin you find.   Domain = {1.00, 0.50, 0.25, 0.10, 0.05, 0.01}, Size of the domain=6.

- Domain of a random variable can be "countably infinite."

Example: X = number of words in the next Game of Thrones novel. No matter how large you guess, it's possible it might be even longer, so we say the domain is infinite.

Requirement: $1 = \text{sum } P(X=x)$

- Domain of a random variable can be "uncountably infinite."

Example: a variable whose value can be ANY REAL NUMBER.

How we deal with this: $P(X=x)$ is ill-defined, but $P(a \leq X < b)$ is well-defined.

Expectation and PMF

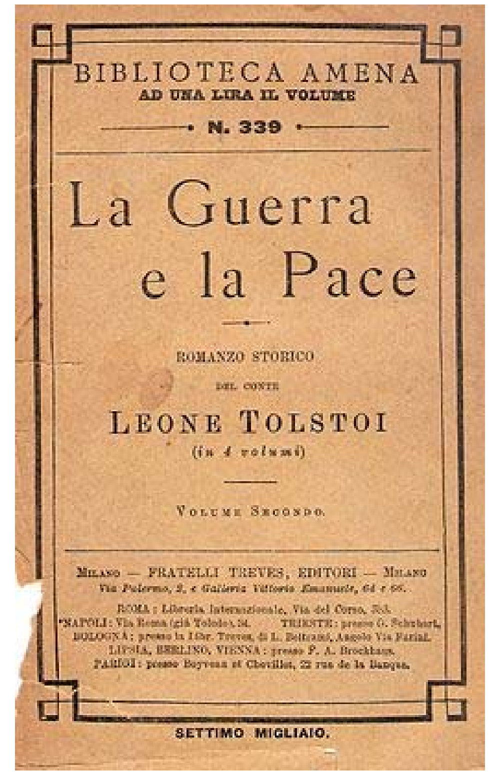
- Expected value can be calculated from PMF only if the domain is finite, or countably infinite.

E[X] = sum value * P(X=value)

Example: X = number of words in the next GoT novel.

E[X] = P(X=1) + 2*P(X=2) + 3*P(X=3) + …

If you know P(X=x) for all x (even if "all x" is an infinite set), then you can compute this expectation by solving the infinite series.



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Joint Random Variables

Jointly distributed random variables (W,X,Y,Y) have a joint pmf, which specifies the probability P(W=w,X=x,Y=y,Z=z) for every tuple of values (w,x,y,z).

Example: W=red, X=purple, Y=green, Z=blue

w	x	y	z	P(W=w,X=x,Y=y,Z=z)
1	1	1	1	1/1296
1	1	1	2	1/1296
…		…		…
6	6	6	4	1/1296
6	6	6	5	1/1296
6	6	6	6	1/1296

Marginalization

$$P(X = x) = \sum_{w} \sum_{y} \sum_{z} P(W = w, X = x, Y = y, Z = z)$$

Example: if W, X, Y, Z are four independent dice, then the marginal is just what you would expect:

$$P(X = x) = \sum_{w=1}^{6} \sum_{y=1}^{6} \sum_{z=1}^{6} \left(\frac{1}{1296}\right) = \frac{1}{6}$$

Conditioning

$$P(X = x|Z = z) = \frac{P(X = x, Z = z)}{P(Z = z)}$$

Example: if W, X, Y, Z are four independent dice, then the marginal is just what you would expect:

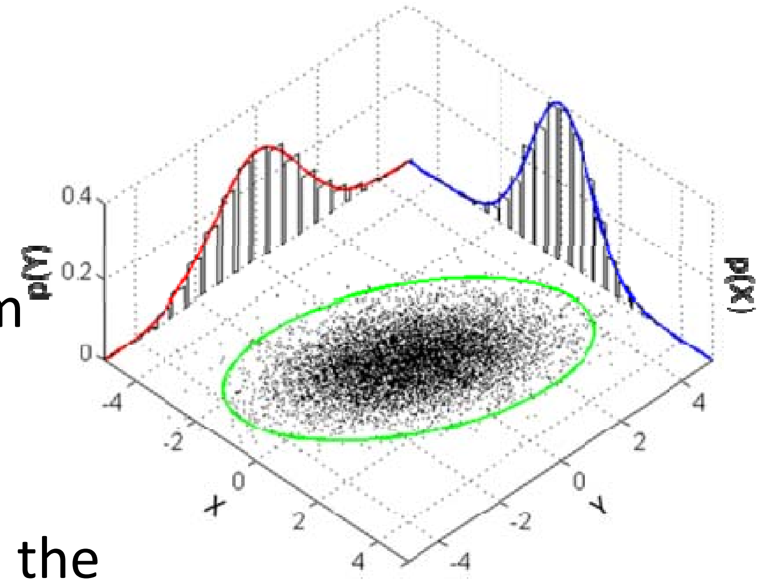
$$P(X = 3|Z = 3) = \frac{P(X = 3, Z = 3)}{P(Z = 3)} = \frac{1/36}{1/6} = \frac{1}{6}$$

Law of total probability

$$P(X=x)=\sum_{i=1}^{n}P(X=x,Y=y_i)$$

$$=\sum_{i=1}^{n}P(X=x\mid Y=y_i)P(Y=y_i)$$

Random Vector



A Random Vector, $\vec{X}$, is a vector of joint random variables $\vec{X} = [X_1, X_2, \ldots, X_n]$.

The PMF of the random vector is defined to be the Joint PMF of all of its component variables:

$$P(\vec{X} = \vec{x}) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

Jointly Random Class and Measurement Variables

The most important case of joint random variables for AI: jointly random categorical (class) and numerical (measurement) variables.

For example, Y= type of fruit, X = weight of the fruit.

x	y	P(X=x,Y=y)
10g	Grape	0.68
10g	Apple	0.06
100g	Grape	0.02
100g	Apple	0.34

We'll talk A LOT more about this in the next lecture (Bayesian inference).

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Functions of Random Variables: PMF

The PMF for a function of random variables is computed the same way as any other marginal: by adding up the component probabilities.

Example: S = W+X+Y+Z

w	x	y	z	s	P(W=w,X=x,Y=y,Z=z,S=s)
1	1	1	1	4	1/1296
1	1	1	2	5	1/1296
1	1	2	1	5	1/1296
…	…	…	…	…	…

s	P(S=s)
4	1/1296
5	4/1296
…	…

Functions of Random Variables: PMF

Example: A = X^2

w	a	P(A=a)
1	1	1/6
2	4	1/6
3	9	1/6
4	16	1/6
5	25	1/6
6	36	1/6

Functions of Random Variables: Expectation

It's important to know that, for any function g(X), $E[g(X)] \neq g(E[X])$

$$E[g(X)] = \sum_{g} g * P(g(X) = g)$$

$$g(E[X]) = g\left(\sum_{x} x * P(X = x)\right)$$

Those are not the same thing!!

Functions of Random Variables: Expectation

Example: $E[X^2] \neq E[X]^2$

$$E[X^2] = 1\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + \cdots + 36\left(\frac{1}{6}\right) = 15.1667$$

$$E[X]^2 = (3.5)^2 = 12.25$$

Those are not the same thing!!

Summary: Random Variables

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"ABOUT THIS EXPERIMENT FOR GENERATING RANDOM NUMBERS – EACH TIME YOU DO IT, IT COMES OUT DIFFERENT."