CS440/ECE448 Lecture 17: Bayesian Inference

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Review: Probability

- Random variables, events
- Axioms of probability
- Atomic events
- Joint and marginal probability distributions
- Conditional probability distributions
- Product rule, chain rule
- Independence and conditional independence

Outline: Bayesian Inference

- Bayes Rule
- Law of Total Probability
- Misdiagnosis
- MAP = MPE
- The “Naïve Bayesian” Assumption
- Bag of Words (BoW)
- Parameter Estimation for the BoW model

Bayes Rule

Rev. Thomas Bayes
(1702-1761)

- The product rule gives us two ways to factor a joint probability:

$$P(A, B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

- Therefore,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- Why is this useful?
  - Can update our beliefs about A based on evidence B
    - P(A) is the *prior* and P(A|B) is the *posterior*
  - Key tool for probabilistic inference: can get *diagnostic probability* from *causal probability*
    - E.g., P(Cavity = true | Toothache = true) from P(Toothache = true | Cavity = true)

Bayes Rule example

Dan & Dana are getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year (5/365 = 0.014). Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on their wedding?

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$$P(rain|forecast) = \frac{0.9 * 0.014}{P(forecast)}$$

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$$P(rain|forecast) = \frac{0.9 \times 0.014}{0.9 * 0.014 + 0.1 * 0.986}$$

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Law of total probability

$$P(X = x) = \sum_{i=1}^{n} P(X = x, Y = y_i)$$

$$= \sum_{i=1}^{n} P(X = x \mid Y = y_i) P(Y = y_i)$$

Bayes Rule example

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$$P(\text{rain} \mid \text{predict}) = \frac{P(\text{predict} \mid \text{rain})P(\text{rain})}{P(\text{predict})}$$

$$= \frac{P(\text{predict} \mid \text{rain})P(\text{rain})}{P(\text{predict} \mid \text{rain})P(\text{rain}) + P(\text{predict} \mid \neg\text{rain})P(\neg\text{rain})}$$

$$= \frac{0.9 \times 0.014}{0.9 \times 0.014 + 0.1 \times 0.986} = \frac{0.0126}{0.0126 + 0.0986} = 0.111$$

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Bayes rule: Example

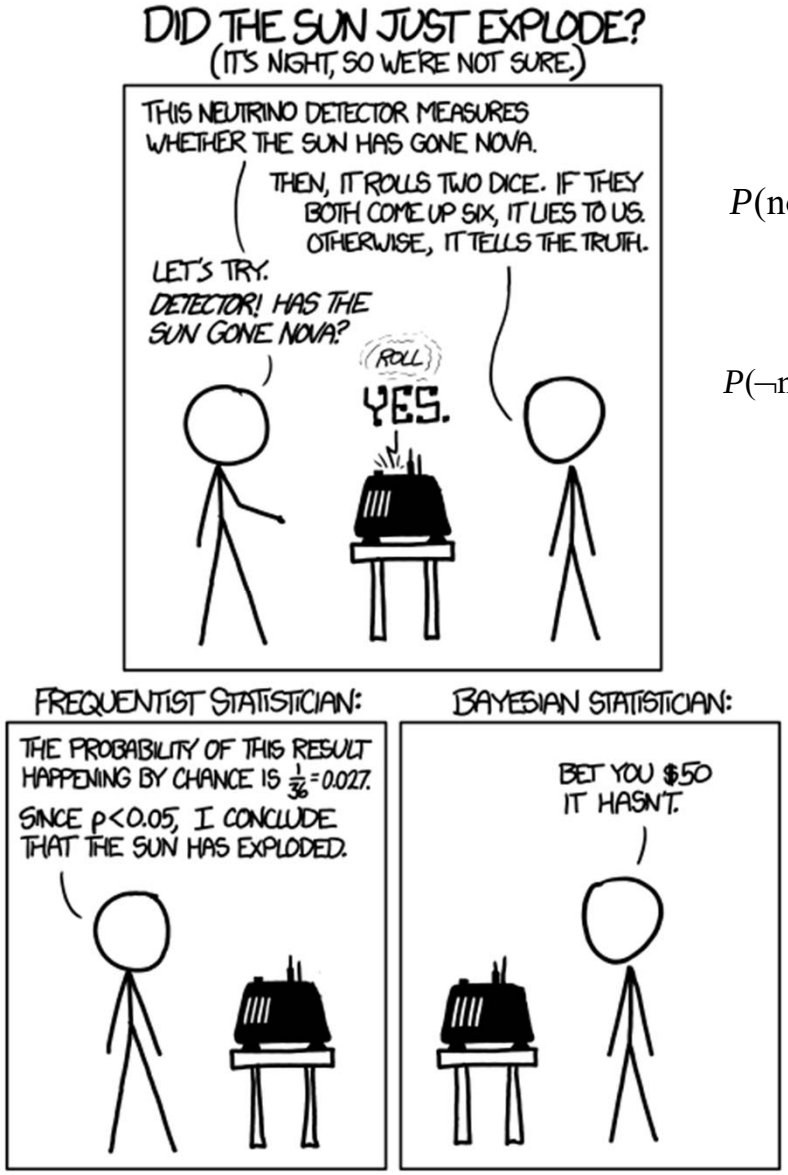
1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get positive mammographies. 9.6% of women without breast cancer will also get positive mammographies. A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer?

$$P(\text{cancer} \mid \text{positive}) = \frac{P(\text{positive} \mid \text{cancer})P(\text{cancer})}{P(\text{positive})}$$

$$= \frac{P(\text{positive} \mid \text{cancer})P(\text{cancer})}{P(\text{positive} \mid \text{cancer})P(\text{cancer}) + P(\text{positive} \mid \neg\text{cancer})P(\neg\text{Cancer})}$$

$$= \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.096 \times 0.99} = \frac{0.008}{0.008 + 0.095} = 0.0776$$

https://www.youtube.com/watch?v=BcvLAw-JRss



$$P(\text{nova} \mid \text{yes}) = \frac{P(\text{yes} \mid \text{nova})P(\text{nova})}{P(\text{yes})}$$

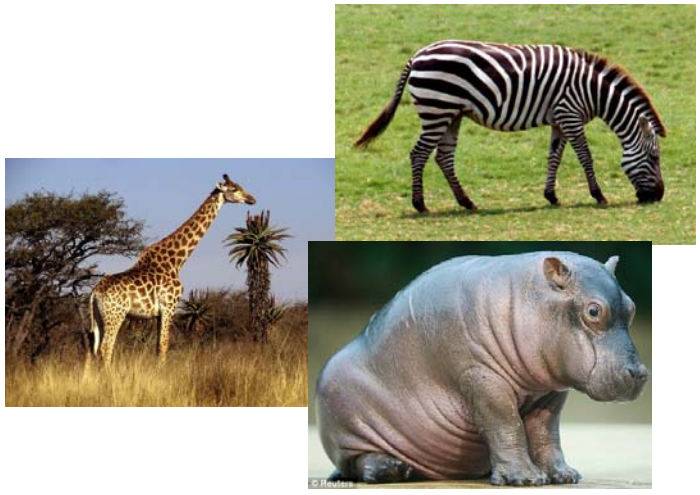
$$P(\neg\text{nova} \mid \text{yes}) = \frac{P(\text{yes}|\neg\text{nova})P(\neg\text{nova})}{P(\text{yes})}$$

https://xkcd.com/1132/

See also: https://xkcd.com/882/

Probabilistic inference

- Suppose the agent has to make a decision about the value of an unobserved *query variable* X given some observed *evidence variable(s)* E = e
  - Partially observable, stochastic, episodic environment
  - Examples: X = {spam, not spam}, e = email message
    X = {zebra, giraffe, hippo}, e = image features



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Bayesian decision theory

- Let x be the value predicted by the agent (known) and x* be the true value of X (unknown).
- The agent has a **loss function**, which is 0 if x = x* and 1 otherwise
- Expected loss for predicting x:

$$\sum_{x^*} L(x, x^*) P(x^*|o) = \sum_{x \neq x^*} P(x^*|o) = 1 - P(x|o)$$

- What is the estimate of X that minimizes the expected loss?
  - The one that has the greatest posterior probability P(x|o)
  - This is called the Maximum a Posteriori (MAP) decision

MAP decision

- Value x of X that has the highest posterior probability given the observation O=o:

$$x^* = \text{argmax} P(x|o) = \text{argmax} \frac{P(o|x)P(x)}{P(o)}$$
$$= \text{argmax} P(o|x)P(x)$$

$$P(x \mid e) \propto P(e \mid x)P(x)$$

posterior likelihood prior

- Maximum Likelihood (ML) decision:

$$x^* = \text{argmax} P(o|x)$$

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Naïve Bayes model

- Suppose we have many different types of observations (symptoms, features) $E_1, \ldots, E_n$ that we want to use to obtain evidence about an underlying hypothesis X
- MAP decision:

$$P(X = x \mid E_1 = e_1, \ \ldots, E_n = e_n)$$
$$\propto P(X = x)P(E_1 = e_1, \ \ldots, E_n = e_n \mid X = x)$$

  - If each feature E_i can take on k values, how many entries are in the (conditional) joint probability table $P(E_1, \ldots, E_n \mid X = x)$?

Naïve Bayes model

- Suppose we have many different types of observations (symptoms, features) $E_1, \ldots, E_n$ that we want to use to obtain evidence about an underlying hypothesis X
- MAP decision:

$$P(X = x \mid E_1 = e_1, \ \ldots, E_n = e_n)$$

$$\propto P(X = x)P(E_1 = e_1, \ \ldots, E_n = e_n \mid X = x)$$

- We can make the simplifying assumption that the different features are conditionally independent *given the hypothesis*:

$$P(E_1 = e_1, \ \ldots, E_n = e_n \mid X = x) = \prod_{i=1}^{n} P(E_i = e_i \mid X = x)$$

  - If each observation and the hypothesis can take on *k* values, what is the complexity of storing the resulting distributions?
  - W.o naïve Bayes: $k(k^n - 1)$
  - With naïve Bayes: each p(e|x) requires (k-1)*k (k values of x, k-1 of e)
  - There are n of them -> n*(k-1)*k

Naïve Bayes model

- Posterior:

$$P(X = x \mid E_1 = e_1, \ \ldots, E_n = e_n)$$

- MAP decision:

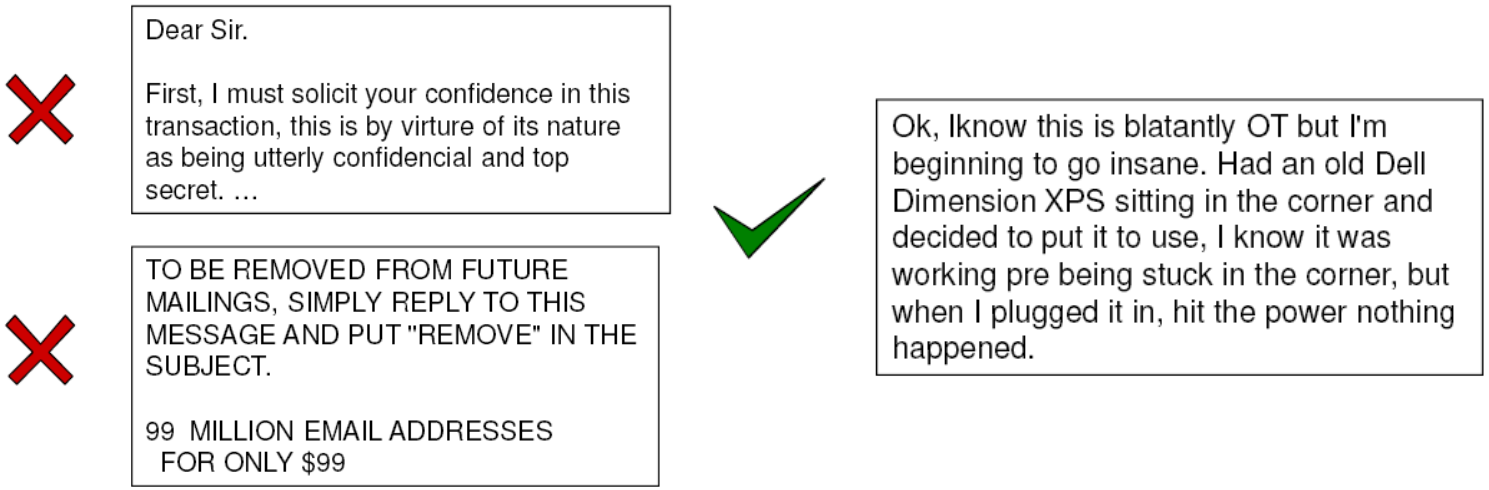
$$x^* = \arg\max_x P(x \mid e) \propto P(x) \prod_{i=1}^{n} P(e_i \mid x)$$

posterior prior likelihood

Case study:
Text document classification

- **MAP decision:** assign a document to the class with the highest posterior P(class | document)

- Example: spam classification
  - Classify a message as spam if P(spam | message) > P(¬spam | message)



Case study:
Text document classification

- **MAP decision:** assign a document to the class with the highest posterior P(class | document)

- We have P(class | document) $\propto$ P(document | class)P(class)

- To enable classification, we need to be able to estimate the likelihoods P(document | class) for all classes and
priors P(class)

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Naïve Bayes Representation

- Goal: estimate likelihoods P(document | class) and priors P(class)
- Likelihood: ***bag of words*** representation
  - The document is a sequence of words (w_1, …, w_n)
  - The order of the words in the document is not important
  - Each word is conditionally independent of the others given document class

Dear Sir.

First, I must solicit your confidence in this transaction, this is by virture of its nature as being utterly confidencial and top secret. …

TO BE REMOVED FROM FUTURE MAILINGS, SIMPLY REPLY TO THIS MESSAGE AND PUT "REMOVE" IN THE SUBJECT.

99 MILLION EMAIL ADDRESSES FOR ONLY $99

Ok, Iknow this is blatantly OT but I'm beginning to go insane. Had an old Dell Dimension XPS sitting in the corner and decided to put it to use, I know it was working pre being stuck in the corner, but when I plugged it in, hit the power nothing happened.

Naïve Bayes Representation

- Goal: estimate likelihoods P(document | class) and priors P(class)
- Likelihood: ***bag of words*** representation
  - The document is a sequence of words (w_1, …, w_n)
  - The order of the words in the document is not important
  - Each word is conditionally independent of the others given document class

$$P(document \mid class) = P(w_1, \ \ldots, w_n \mid class) = \prod_{i=1}^{n} P(w_i \mid class)$$

Bag of words illustration

US Presidential Speeches Tag Cloud
http://chir.ag/projects/preztags/

Bag of words illustration

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Naïve Bayes Representation

- Goal: estimate likelihoods P(document | class) and P(class)
- Likelihood: ***bag of words*** representation
  - The document is a sequence of words (w_1, … , w_n)
  - The order of the words in the document is not important
  - Each word is conditionally independent of the others given document class

$$P(document \mid class) = P(w_1, \ \ldots, w_n \mid class) = \prod_{i=1}^{n} P(w_i \mid class)$$

  - Thus, the problem is reduced to estimating marginal likelihoods of individual words $P(w_i \mid class)$

Parameter estimation

- Model parameters: feature likelihoods P(word | class) and priors P(class)
  - How do we obtain the values of these parameters?

prior

spam:	0.33
¬spam:	0.67

P(word | spam)

the :	0.0156
to :	0.0153
and :	0.0115
of :	0.0095
you :	0.0093
a :	0.0086
with:	0.0080
from:	0.0075
...	

P(word | ¬spam)

the :	0.0210
to :	0.0133
of :	0.0119
2002:	0.0110
with:	0.0108
from:	0.0107
and :	0.0105
a :	0.0100
...	

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Parameter estimation

- Model parameters: feature likelihoods P(word | class) and priors P(class)
  - How do we obtain the values of these parameters?
  - Need *training set* of labeled samples from both classes

$$P(\text{word} \mid \text{class}) = \frac{\#\text{ of occurrences of this word in docs from this class}}{\text{total \# of words in docs from this class}}$$

  - This is the *maximum likelihood* (ML) estimate, or estimate that maximizes the likelihood of the training data:

$$\prod_{d=1}^{D} \prod_{i=1}^{n_d} P(w_{d,i} \mid class_{d,i})$$

d: index of training document, *i*: index of a word

Parameter estimation

- ML (Maximum Likelihood) parameter estimate:

$$P(\text{word} \mid \text{class}) = \frac{\#\text{ of occurrences of this word in docs from this class}}{\text{total \# of words in docs from this class}}$$

- Laplacian Smoothing estimate
  - How can you estimate the probability of a word you never saw in the training set? (Hint: what happens if you give it probability 0, then it actually occurs in a test document?)
  - **Laplacian smoothing:** pretend you have seen every vocabulary word one more time than you actually did

$$P(\text{word} \mid \text{class}) = \frac{\#\text{ of occurrences of this word in docs from this class} + 1}{\text{total \# of words in docs from this class} + V}$$

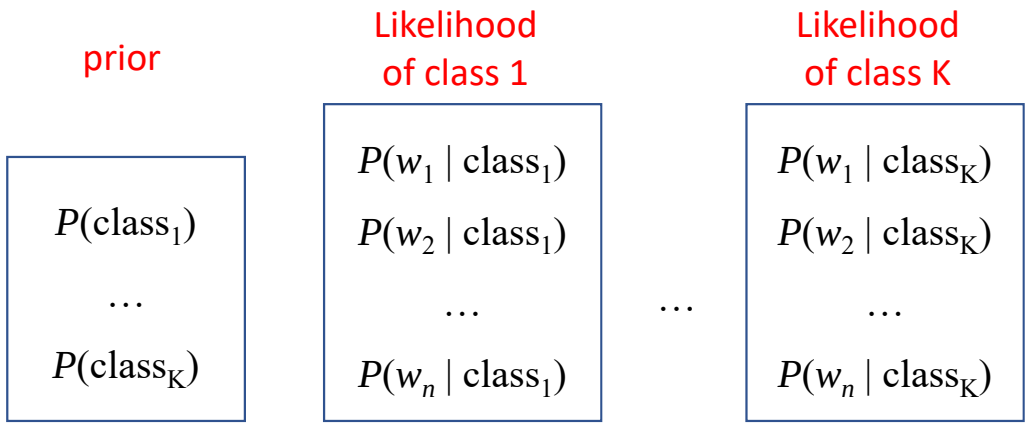
(V: total number of unique words)

Summary: Naïve Bayes for Document Classification

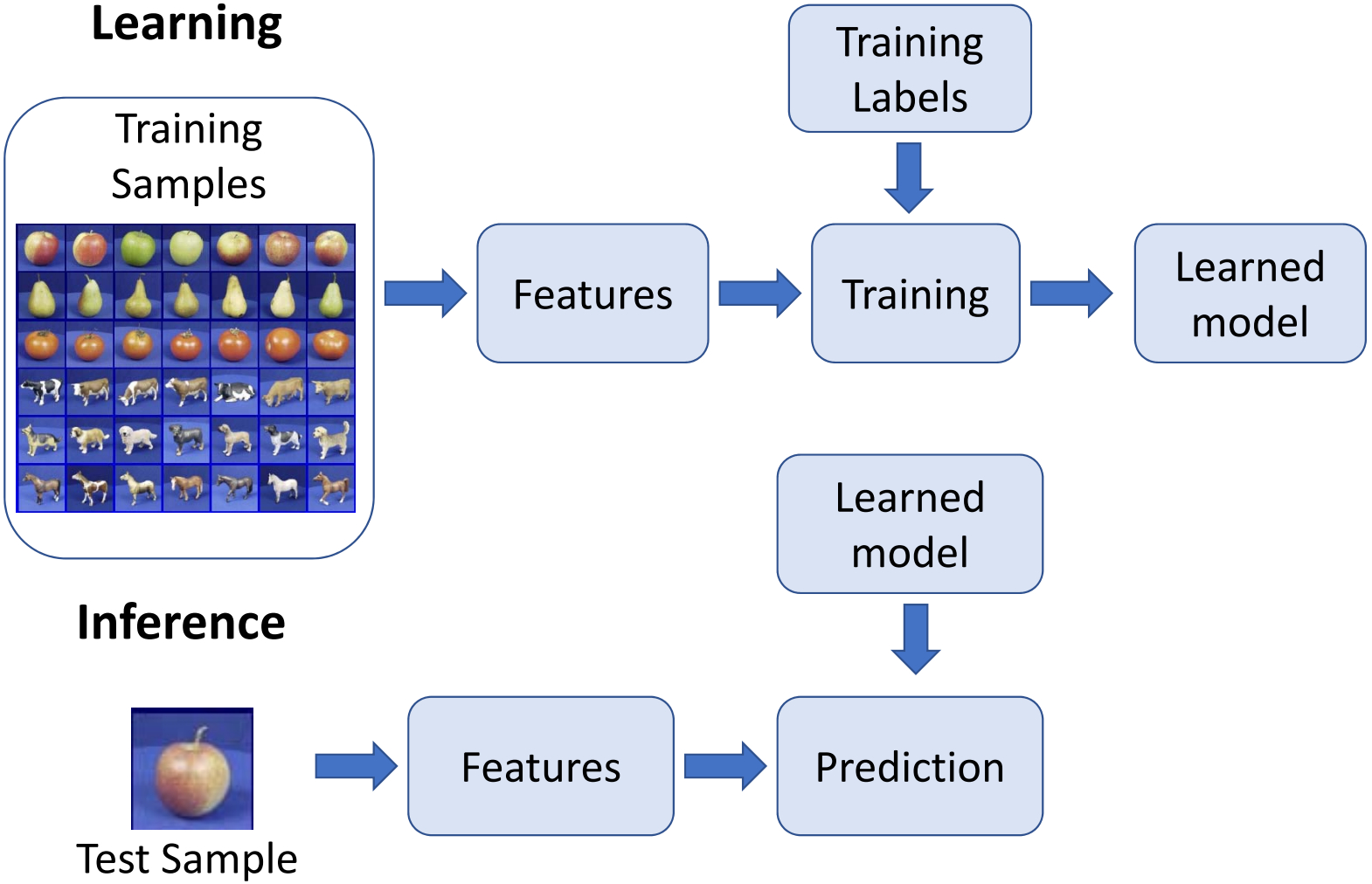
- Naïve Bayes model: assign the document to the class with the highest posterior

$$P(class \mid document) \propto P(class) \prod_{i=1}^{n} P(w_i \mid class)$$

- Model parameters:



Learning and inference pipeline



Review: Bayesian decision making

- Suppose the agent has to make decisions about the value of an unobserved *query variable* X based on the values of an observed *evidence variable* E
- **Inference problem:** given some observation E = e, what is P(X | e)?
- **Learning problem:** estimate the parameters of the probabilistic model P(X | E) given a *training sample* $\{(x_1,e_1), \ldots, (x_n,e_n)\}$