

ECE/CS 434 : Real-World Algorithms for IoT and Data Science :  
Homework 1 Due 11 :59pm, Mon, Feb 17, 2025

Please answer each of the 5 questions on a new page, then scan and upload these pages onto Gradescope.

**Problem 1 : State True/False with a 1 line justification [5x4=20 points]**

(a)  $A$  is a  $m \times n$  matrix with  $m < n$ . The null space  $N(A)$  is always 0.

(b) The matrix  $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , when left multiplied with a second matrix  $B$  (i.e.  $B * A$ ), subtracts twice of the first column of  $B$  from the second column of  $B$ .

(c) For an orthonormal matrix  $Q$  (i.e., columns of  $Q$  are orthogonal to each other and the length of each column is 1),  $Q^{-1} = Q^T$ .

(d) If  $b_1, b_2, b_3$  form the basis of a space, then  $c_1 b_1 + c_2 b_2 + c_3 b_3 = 0$  implies that all  $c_1 = c_2 = c_3 = 0$ .

(e) Matrix  $A$  has 6 columns, each column being a 10 dimensional vector. You are told that the dimension of  $N(A^T)$  is 5. Then,  $N(A)$  must be 1 and  $\text{Rank}(A)$  must be 5.

**Problem 2 : Symmetric Matrices**

**[10+10=20 points]**

(a) Prove that  $A^T A$  is a symmetric matrix.

*Hint* : use the basic properties of transpose, as discussed in class.

(b) Prove that  $\text{Rank}(AB) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}$

**Problem 3 : Column Spaces**

**[10+10=20 points]**

(a) Choose  $b$  which gives no solution and another  $b$  which gives infinitely many solutions. Your answer should show two values of  $b$ .

$$3x + 2y = 10 \tag{1}$$

$$6x + 4y = b \tag{2}$$

(b) Consider matrix  $A_{m \times n}$ . You are told  $r = \text{Rank}(A)$  and  $r < m$  and  $r < n$ . How many solutions are possible for the equation  $Ax = b$ ? What is the dimensions of  $N(A)$ ?

## Problem 4 : Least Squares

[10 points]

Consider the following system of equations (called an over-determined system since there are more equations than unknowns) :

$$x - y = 2 \quad (3)$$

$$x + y = 4 \quad (4)$$

$$2x + y = 8 \quad (5)$$

How many solutions exist for the above system of equations? If a solution exists find one, if not, determine the least squares solution for  $x$  and  $y$ .

## Problem 5 : Eigenvalues and Eigenvectors [5+5+5+5+10=30 points]

(a) Prove that, for symmetric matrix  $A$ , eigenvalues of matrix  $A^2 = (\text{Eigenvalue of matrix } A)^2$

(b) Prove that  $\lambda(A - \sigma I) = (\lambda(A) - \sigma)$  where  $\lambda(M)$  denotes the eigenvalues of matrix  $M$ ,  $I$  is the identity matrix, and  $\sigma$  is an arbitrary constant.

(c) Given a matrix  $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ , calculate its eigen vectors  $e_1$ ,  $e_2$ , and  $e_3$ . Choose one eigen vector

$e$ , plot  $e$  and  $e' = A \cdot e$  in the 3D space. Consider another vector  $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ . Plot  $x$  and  $x' = A \cdot x$  in the 3D space.

(d) For the scenario above, write TRUE or FALSE :

- (i) if  $e$  and  $e'$  lie on the same line ;
- (ii) if  $x$  and  $x'$  lie on the same line.

(e) Consider the same matrix  $A$  as above. Use Python to plot the following points in 3D space.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Use another graph to plot their locations in a new 3D space where the basis of this new 3D space are the Eigen vectors of  $A$ . Please identify the points whose representation in the new space are unit vectors along  $x$ ,  $y$  or  $z$  directions. Explain the relationship between these vectors and the Eigen vectors.