ECE/CS 434 : Real-World Algorithms for IoT and Data Science : Homework 1 Due 11 :59pm, Mon, Feb 12, 2024

Please answer each of the 5 questions on a new page, then scan and upload these pages onto Gradescope.

Problem 1 : State True/False with a 1 line justification [5x4=20 points]

(a) A is a $m \times n$ matrix with $m < n$. The null space $N(A)$ is always 0.

(b) The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, when left multiplied with a second matrix B (i.e. $B * A$), subtracts twice of the first column of B from the second column of B.

(c) For an orthonormal matrix Q (i.e., columns of Q are orthogonal to each other and the length of each column is 1), $Q^{-1} = Q^T$.

(d) If b_1, b_2, b_3 form the basis of a space, then $c_1b_1 + c_2b_2 + c_3b_3 = 0$ implies that all $c_1 = c_2 = c_3 = 0$.

(e) Matrix A has 6 columns, each column being a 10 dimensional vector. You are told that the dimension of $N(A^T)$ is 5. Then, $N(A)$ must be 1 and Rank(A) must be 5.

Problem 2 : Symmetric Matrices [10+10=20 points]

(a) Prove that $A^T A$ is a symmetric matrix.
Hint : use the basic properties of transpose, as discussed in class.

(b) Prove that $Rank(AB) \leq min\{Rank(A), Rank(B)\}$

Problem 3 : Column Spaces [10+10=20 points]

(a) Choose b which gives no solution and another b which gives infinitely many solutions. Your answer should show two values of b.

$$3x + 2y = 10 \quad (1)$$

$$6x + 4y = b \quad (2)$$

(b) Consider matrix $A_{m \times n}$. You are told $r = Rank(A)$ and $r < m$ and $r < n$. How many solutions are possible for the equation $Ax = b$? What is the dimensions of $N(A)$?

Problem 4 : Least Squares [10 points]

Consider the following system of equations (called an over-determined system since there are more equations than unknowns) :

$$x - y = 2 \tag{3}$$
$$x + y = 4 \tag{4}$$
$$2x + y = 8 \tag{5}$$

How many solutions exist for the above system of equations? If a solution exists find one, if not, determine the least squares solution for x and y.

Problem 5 : Eigenvalues and Eigenvectors [5+5+5+5+10=30 points]

(a) Prove that, for symmetric matrix A, eigenvalues of matrix A^2 = (Eigenvalue of matrix A)^2

(b) Prove that $\lambda\Big(A - \sigma I\Big) = \Big(\lambda(A) - \sigma\Big)$ where $\lambda(M)$ denotes the eigenvalues of matrix M, I is the identity matrix, and σ is an arbitrary constant.

(c) Given a matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, calculate its eigen vectors e_1, e_2, and e_3. Choose one eigen vector e, plot e and $e' = A \cdot e$ in the 3D space. Consider another vector $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Plot x and $x' = A \cdot x$ in the 3D space.

(d) For the scenario above, write TRUE or FALSE :
  (i) if e and e' lie on the same line;
  (ii) if x and x' lie on the same line.

(e) Consider the same matrix A as above. Use Python to plot the following points in 3D space.
$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
Use another graph to plot their locations in a new 3D space where the basis of this new 3D space are the Eigen vectors of A. Please identify the points whose representation in the new space are unit vectors along x, y or z directions. Explain the relationship between these vectors and the Eigen vectors.