

ECE/CS 434 : Mobile Computing Algorithms and Applications :

Homework 2 : Due 11 :59pm, Wed, Mar 1, 2023

Please start each of the 7 questions on a new page, then scan and upload these pages onto Gradescope.

Problem 1 **[15 points]**

Consider the 2^{nd} column of the Fourier matrix, which is $[e^{j0} \ e^{j\theta} \ e^{j2\theta} \ \dots \ e^{j(N-1)\theta}]^T$.

- (a) Prove that the 3^{rd} column is orthogonal to the 2^{nd} column.
- (b) Prove that any column is orthogonal to the 2^{nd} column.
- (c) Prove that any two columns are orthogonal.

Problem 2 **[24 points]**

Consider a N dimensional vector \bar{v} expressed in the identity basis.

- (a) Express the vector \bar{v} in an orthonormal basis F , where F is a $N \times N$ matrix.
(*Hint : See class notes on how we express a signal in different basis.*)
- (b) Let's call the above vector \bar{w} . Now create a matrix B such that $B\bar{w}$ scales the i^{th} element of \bar{w} by a scalar b_i . What should be the matrix B ?
- (c) Let's denote the vector $B\bar{w}$ as vector \bar{z} . Now convert vector \bar{z} back into the original identity basis.
- (d) Now write all the above operations on vector \bar{v} in one equation in terms of F and B .
- (e) Write the Eigen-decomposition equation of a matrix A , where S contains the eigenvectors of A and Λ is the diagonal matrix containing the eigenvalues.
- (f) Given the above exercise you have done, explain in plain English what Eigen-decomposition does to a vector (in other words, what happens when matrix A is multiplied to vector x)?

Problem 3 **[4 points]**

In class, we discussed the analogy of expressing a painting “lonely queen” in two different bases ; one was $\langle \text{red, green, blue} \rangle$ and the other was $\langle \text{purple, brown, white} \rangle$.

Can you come up with another analogy from the real world where the same “thing” can be expressed in 2 different “bases”. Write the “thing” and the 2 “bases”.

Problem 4 **[12 points]**

- (a) You are sampling a signal every 0.25 millisecond. What is the maximum frequency you would be able to see in FFT?

(b) Suppose you take $N = 1000$ for your FFT. At what frequency resolution would you be able to analyze the signal you are sampling? A frequency resolution of R Hz means you express the signal at frequencies $[0, R, 2R, \dots]$ Hz.

(c) True/False : The FFT of any signal is symmetric around frequency zero. Explain your answer in 1 sentence.

(d) Say X_f is the DFT of a signal x_n . Now, consider $Y_f = X_f \cdot e^{j\phi}$, where ϕ is a constant angle. Is the IDFT(Y_f) a shifted version of the signal x_n ? Briefly argue in favor or against.

Problem 5

[10 points]

(a) Consider a signal $x[n] = \cos(2\pi f_1 n t_s) + 2 \sin(2\pi f_1 n t_s) + \sin(4\pi f_1 n t_s)$. Draw the magnitude and phase plots of X_f , which is the DFT of $x[n]$. Assume that f_1 is the fundamental frequency in which you are sampling the signal.

(b) Prove that DFT is linear, i.e., $\text{DFT}(a_1 x[n] + a_2 y[n]) = a_1 X_f + a_2 Y_f$, where X_f and Y_f are the DFTs of $x[n]$ and $y[n]$, respectively.

Problem 6

[5 points]

Use your phone to record your own voice, and say “My name is [Your Name].” Use Python to import the saved audio file, and compute its FFT. Submit the plot of the *magnitude* of the FFT.

Hint : You can use `scipy.io.wavfile.read`¹ to read the audio file and get the sampling rate. If your data has two channels, you can extract 1 with `data = data[:, 0]`. You can then compute the FFT with `scipy.fft`².

Problem 7

[30 points]

(a) Prove that : $P(A, B, C, \dots Z) = P(A|B, C \dots Z)P(B|C, D, \dots Z) \dots P(Z)$

(b) Given two random variables X_1 and X_2 , write an equation to check if they are uncorrelated.

(c) Say data $D_1 = [2, 9, 10, 7, 2, 6, 8, 16, 14]$ and data $D_2 = [4, 12, 14, 8, 5, 7, 11, 16, 10]$.

Are the data uncorrelated?

(d) Given two random variables X_1 and X_2 , write an equation to check if they are independent.

(e) Can two data streams be dependent but uncorrelated?

If so, give an example. If not, explain why not.

(f) Use any programming language to generate 3 sequences, each containing 10 integers generated uniform randomly between $[-10, 10]$. Let's call these sequences S_1, S_2, S_3 and say a vector $S = [S_1, S_2, S_3]^T$. Calculate (using pen and paper, or using a calculator) the covariance matrix $Cov(S)$ and write it out in matrix form.

(g) Random variables Y_1 and Y_2 are jointly Gaussian. Write their joint distribution in terms of their mean vector μ_Y and the covariance matrix Σ_Y .

(h) Roughly sketch the above PDF when :

(I) $\mu_Y = [0, 0]^T$ and $\Sigma_Y = I$, the Identity matrix.

(II) $\mu_Y = [1, 3]^T$ and $\Sigma_Y = I$, the Identity matrix.

1. <https://docs.scipy.org/doc/scipy/reference/generated/scipy.io.wavfile.read.html>

2. <https://docs.scipy.org/doc/scipy/reference/tutorial/fft.html#fast-fourier-transforms>

(III) $\mu_Y = [1, 3]^T$ and $\Sigma_Y \neq I$.

(i) You collect some data as a vector $D = [d_1, d_2, \dots, d_n]^T$. Say $\sigma^2 = \text{Var}(D)$, where Var implies variance. You now multiply each data value with a constant k and then add c to each of them. What is $\text{Var}(kD + c)$ in terms of σ^2 .

(j) A Bivariate Gaussian distribution of X_1 and X_2 has a covariance matrix which is an Identity.

(I) Do you think X_1 and X_2 are independent? Why or why not?

(II) Calculate $P(X_1 < 0.5 | X_2 = 0.9)$, assuming both X_1 and X_2 are mean zero random variables.