Lecture 4: Signal Processing Review

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Outline

Four Fourier Transforms

1. CTFS: Periodic in time, Discrete in frequency
2. CTFT: Periodic in neither, Discrete in neither
3. DTFT: Discrete in time, Periodic in frequency
4. DFT: Discrete in both, Periodic in both

Continuous Time Fourier Series: Periodic in time, Discrete in frequency

Forward Transform:

$$X_k = \frac{1}{T_0}\int_{t=0}^{T_0} x(t)e^{-jk\Omega_0 t}dt, \quad \Omega_0 = \frac{2\pi}{T_0}$$

Inverse Transform:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

Parseval's Theorem:

$$\frac{1}{T_0}\int_{t=0}^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Continuous Time Fourier Transform: Periodic in neither, Discrete in neither

Forward Transform:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Inverse Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

Parseval's Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

Discrete Time Fourier Transform: Discrete in time, Periodic in frequency

Forward Transform:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Inverse Transform:

$$x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega) e^{j\omega n} d\omega$$

Parseval's Theorem:

$$\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |X(\omega)|^2 d\omega$$

Discrete Fourier Transform: Discrete in both, Periodic in both

Forward Transform:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

Inverse Transform:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

Parseval's Theorem:

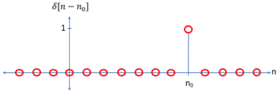
$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Outline

Discrete-Time Impulse

Definition:

$$\delta[n-n_0] = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$$



Sampling Theorem:

$$\sum_{n=-\infty}^{\infty} \delta[n-n_0] f[n] = f[n_0]$$

DTFT of an Impulse:

$$x[n] = \delta[n-n_0] \leftrightarrow X(\omega) = e^{-j\omega n_0}$$

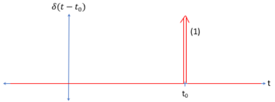
DFT of a Cosine:

$$x[n] = \cos\left(\left(\frac{2\pi a}{N}\right) n\right) \leftrightarrow X[k] = \frac{N}{2}\delta[k-a] + \frac{N}{2}\delta[k-(N-a)]$$

Continuous-Time Impulse

Definition:

$$\delta(t - t_0) = \begin{cases} \infty & t = t_0 \\ 0 & \text{otherwise} \end{cases}$$



Sampling Theorem:

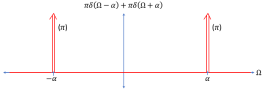
$$\int_{t=-\infty}^{\infty} \delta(t - t_0) f(t) = f(t_0)$$

CTFT of an Impulse:

$$x[n] = \delta(t - t_0) \leftrightarrow X(\Omega) = e^{-j\Omega t_0}$$

CTFT of a Cosine:

$$x(t) = \cos(\alpha t) \leftrightarrow X(\Omega) = \pi\delta(\Omega - \alpha) + \pi\delta(\Omega + \alpha)$$



Outline

DTFT: Rectangle $\leftrightarrow$ Sinc

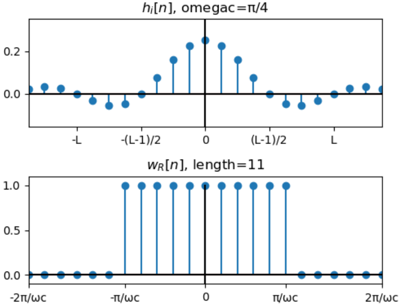
- The DTFT of a sinc is a rectangle:

$$h[n] = \left(\frac{\omega_c}{\pi}\right)\text{sinc}(\omega_c n) \quad \leftrightarrow \quad H(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

- The DTFT of an even-symmetric rectangle is a sinc-like function, called the Dirichlet form:

$$d_L[n] = \begin{cases} 1 & |n| \le \frac{L-1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Rectangle ↔ Sinc



Symmetric and Causal Rectangles

The causal rectangular window is:

$$w_R[n] = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Its DTFT is:

$$\begin{aligned} W_R(\omega) &= \sum_{n=-\infty}^{\infty} w_R[n] e^{-j\omega n} = \sum_{n=0}^{L-1} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} \\ &= \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega\left(\frac{L-1}{2}\right)} \end{aligned}$$

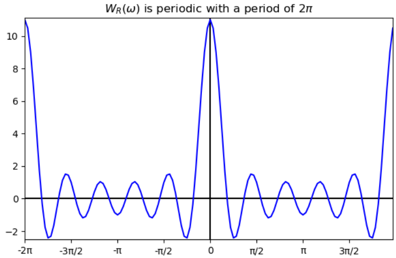
It's just a delayed version of the symmetric rectangle:

$$w_R[n] = d_L\left[n - \frac{L-1}{2}\right] \leftrightarrow W_R(\omega) = D_L(\omega) e^{-j\omega\left(\frac{L-1}{2}\right)}$$

Properties of the Dirichlet form: Periodicity

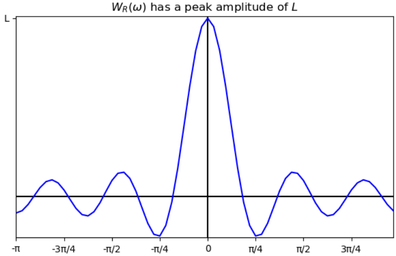
$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Both numerator and denominator are periodic with period 2π.



Properties of the Dirichlet form: DC Value

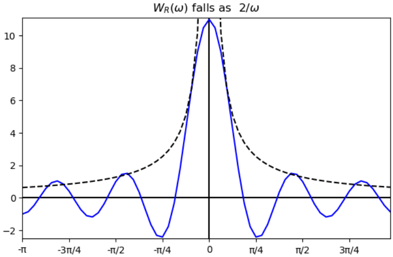
$$D_L(0) = \sum_{n=-\infty}^{\infty} w[n] = L$$



Properties of the Dirichlet form: Sinc-like

$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} \approx \frac{\sin(\omega L/2)}{\omega/2}$$

Because, for small values of ω, $\sin\left(\frac{\omega}{2}\right) \approx \frac{\omega}{2}$.



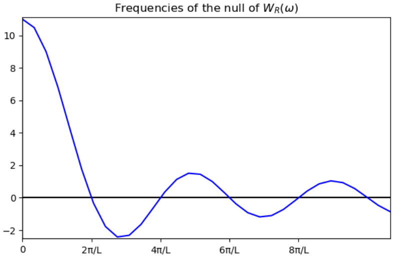
Properties of the Dirichlet form: Nulls

$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

It equals zero whenever

$$\frac{\omega L}{2} = k\pi$$

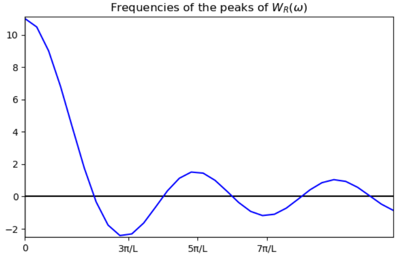
For any nonzero integer, k.



Properties of the Dirichlet form: Sidelobes

Its sidelobes are

$$D_L\left(\frac{3\pi}{L}\right) = \frac{-1}{\sin(3\pi/2L)} \approx \frac{-2L}{3\pi}$$
$$D_L\left(\frac{5\pi}{L}\right) = \frac{1}{\sin(5\pi/2L)} \approx \frac{2L}{5\pi}$$
$$D_L\left(\frac{7\pi}{L}\right) = \frac{-1}{\sin(7\pi/2L)} \approx \frac{-2L}{7\pi}$$
$$\vdots$$



Properties of the Dirichlet form: Relative Sidelobe Amplitudes

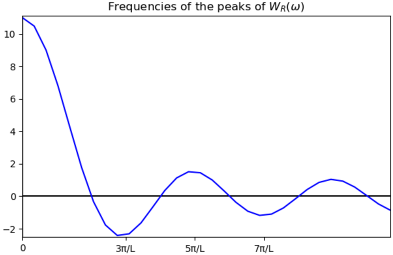
The **relative** sidelobe amplitudes don't depend on L:

$$\frac{D_L\left(\frac{3\pi}{L}\right)}{D_L(0)} = \frac{-1}{L\sin(3\pi/2L)} \approx \frac{-2}{3\pi}$$

$$\frac{D_L\left(\frac{5\pi}{L}\right)}{D_L(0)} = \frac{1}{L\sin(5\pi/2L)} \approx \frac{2}{5\pi}$$

$$\frac{D_L\left(\frac{7\pi}{L}\right)}{D_L(0)} = \frac{-1}{L\sin(7\pi/2L)} \approx \frac{-2}{7\pi}$$

$$\vdots$$



Properties of the Dirichlet form: Decibels

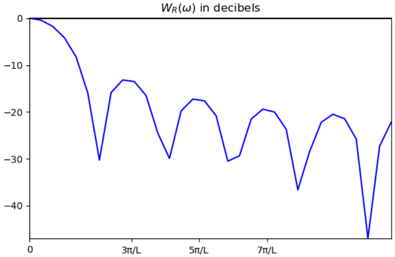
We often describe the relative sidelobe amplitudes in decibels, which are defined as

$$20\log_{10}\left|\frac{D_L\left(\frac{3\pi}{L}\right)}{W(0)}\right| \approx 20\log_{10}\frac{2}{3\pi} \approx -13\text{dB}$$

$$20\log_{10}\left|\frac{D_L\left(\frac{5\pi}{L}\right)}{W(0)}\right| \approx 20\log_{10}\frac{2}{5\pi} \approx -18\text{dB}$$

$$20\log_{10}\left|\frac{D_L\left(\frac{7\pi}{L}\right)}{W(0)}\right| \approx 20\log_{10}\frac{2}{7\pi} \approx -21\text{dB}$$

$$\vdots$$



Outline

Filtering and Windowing

- Filtering: Convolution in time $\leftrightarrow$ Multiplication in frequency
- Windowing: Multiplication in time $\leftrightarrow$ Convolution in frequency

Filtering: Convolution in time $\leftrightarrow$ Multiplication in frequency

$$\begin{aligned} Y(\omega) = H(\omega)X(\omega) \leftrightarrow y[n] &= h[n] * x[n] \\ &= \sum_m h[m]x[n-m] \\ &= \sum_m h[n-m]x[m] \end{aligned}$$

Windowing: Multiplication in time $\leftrightarrow$ Convolution in frequency

$$
\begin{aligned}
y[n] = w[n]x[n] \leftrightarrow Y(\omega) &= \frac{1}{2\pi} W(\omega) * X(\omega) \\
&= \frac{1}{2\pi} \int_0^{2\pi} W(\theta) X(\omega - \theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} W(\omega - \theta) X(\theta) d\theta
\end{aligned}
$$

Outline

Conclusion

1. Periodic in time $\leftrightarrow$ Discrete in frequency
2. Discrete in time $\leftrightarrow$ Periodic in frequency
3. Impulse in time $\leftrightarrow$ Complex exponential in frequency
4. Complex exponential in time $\leftrightarrow$ Impulse in frequency
5. Sinc in time $\leftrightarrow$ Rectangle in frequency
6. Rectangle in time $\leftrightarrow$ Sinc in frequency