ECE 417 Multimedia Signal Processing
Homework 5

UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering

Assigned: Wednesday, 11/3/2021; Due: Tuesday, 11/9/2021

Problem 5.1

Let A be a 2×2 matrix, and let x be one of its elements. All of its other elements are known, and are given as:

$$A = \begin{bmatrix} x & 3 \\ -1 & 2 \end{bmatrix} \tag{5.1-1}$$

Suppose that you are given one of its eigenvalues, λ, and you want to find the corresponding eigenvector. As you know, the scale of an eigenvector is arbitrary, so let's arbitrarily set its first element to 1: $\vec{v} = [1, v_2]^T$. Solve for its second element, v_2, in terms of λ.

Problem 5.2

Suppose $\vec{X} = [X_1, X_2]^T$ is a Gaussian random vector with mean and covariance given by

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Sketch the set of points such that $f_{\vec{X}}(\vec{x}) = \frac{1}{2\pi\sqrt{15}} e^{-\frac{1}{2}}$, where $f_{\vec{X}}(\vec{x})$ is the pdf of $\vec{X}$. Specify four points that are included in this set of points.

Problem 5.3

Principal component analysis finds the eigenvectors of $\boldsymbol{X}\boldsymbol{X}^T$ with maximum eigenvalue, where $\boldsymbol{X}$ is the matrix whose columns are the mean-subtracted data vectors, $\boldsymbol{x}_i - \boldsymbol{\mu} \forall i \in \{1, \ldots, n\}$. Some authors prefer to define PCA without subtracting the mean vector, i.e., directly defining $\tilde{\boldsymbol{X}} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$.

(a) Write $\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}^T$ in terms of $\boldsymbol{X}\boldsymbol{X}^T$, n, and $\boldsymbol{\mu}$. Take advantage of the fact that

$$\boldsymbol{\mu} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i$$

to reduce your answer to only two terms. Hint: if you get stuck, review the proof of the basic probability fact that, for any scalar random variable X, $E[X^2] = \sigma_X^2 + \mu_X^2$. This is just the vector version of that.

(b) What is likely to be the first eigenvector of $\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}^T$? What is the relationship between its other eigenvectors and the eigenvectors of $\boldsymbol{X}\boldsymbol{X}^T$?