ECE 417 Multimedia Signal Processing
Homework 1

UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering

Assigned: Tuesday, 8/22/2023; Due: Tuesday, 8/29/2023

Problem 1.1

According to a story that may or may not be true (`https://en.wikipedia.org/wiki/Galileo%27s_Leaning_Tower_of_Pisa_experiment`), Galileo Galilei dropped two balls of different weights from the leaning tower of Pisa in order to show that they would fall at the same rate. Suppose that you are Galileo's research assistant, and he requires you to determine exactly where the two balls will land. You have available to you precise measurements of three vectors: $x = [x_1, x_2, x_3]^T$ is the position of the top of the tower, $a_1 = [a_{1,1}, a_{1,2}, a_{1,3}]^T$ is the position of the nearest church, and $a_2 = [a_{2,1}, a_{2,2}, a_{2,3}]^T$ is the position of the nearest tavern, all measured with respect to the base of the tower. Your goal is to determine $y = [y_1, y_2, y_3]^T$, the location of the place where Galileo's two weights will hit the ground. You may assume that the ground in Pisa is perfectly flat. The vectors a_1 and a_2 are not parallel, but they are also not orthogonal; the tavern is roughly northwest, while the church is due west. Galileo proposes to you that you may solve this problem using the standard formula for projection of a vector onto a subspace:

$$y = A^T(AA^T)^{-1}Ax \tag{1.1-1}$$

where $A = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \in \Re^{2\times 3}$ is a matrix with a_1^T and a_2^T as its rows.

(a) Prove that the value of y given in Eq. (1.1-1) is unique in that the drop vector, $x - y$, is perpendicular to the ground. In other words, prove that $x - y$ is perpendicular to both a_1 and a_2. In other words, prove that $A(x - y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) Consider the two dot-products, $z_1 = a_1^T x$ and $z_2 = a_2^T x$, i.e., $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Ax$. These represent the distance that the tower top leans in the direction of the church and of the tavern, respectively: to be more precise, $z_1/\|a_1\|$ is the distance that the tower top leans toward the church, and $z_2/\|a_2\|$ is the distance that it leans toward the tavern. Your arch-nemesis, Evil William, seeks to ruin your reputation with Galileo by claiming that y is the wrong distance from the base of the tower. He claims that there is some other vector, $v = y + w$, which is displaced from the bottom of the tower by exactly the same lateral displacement as x, i.e., Evil William claims that $Av = Ax$. Prove that Evil William's claim is false unless the vector w is either zero, or perpendicular to both a_1 and a_2.

(c) Suppose that $A \in \Re^{m\times n}$ is a full-rank short fat matrix ($m < n$ and the rank of A is m), and $z \in \Re^m$ is a vector. Since $m < n$, there are an infinite number of different values of x that could solve the equation $z = Ax$. Prove that the solution $y = A^\dagger z$ is a solution, i.e., $Ay = z$, and is the minimum-norm solution, i.e., $\|y\| < \|x\|$ for any $x \neq y$ such that $z = Ax$. For a full-rank matrix such that $m < n$, the pseudo-inverse may be written as $A^\dagger = A^T(AA^T)^{-1}$.

Problem 1.2

Suppose that $A \in \Re^{m \times n}$ is a tall thin matrix ($m > n$) with full column rank. Consider the equation $Av = b$. In general, since $m > n$, there is no exact solution to this equation. An approximate solution, $v = A^{\dagger}b$, is provided by the pseudo-inverse $A^{\dagger} = (A^T A)^{-1} A^T$ is its pseudo-inverse. Show that v is the minimum-squared error solution to the equation $Av \approx b$, i.e., show that v minimizes

$$E = \|Av - b\|_2^2$$