UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Spring 2021

EXAM 1

Tuesday, September 28, 2021

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 9:30am; you will need to photograph and upload your answers by exactly 11:00am.
- There are a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.
- The last page is a formula sheet, which you may tear off, if you wish.

Name: ____

1. (20 points) So far, we have worked almost exclusively with real-valued signals, but this problem will work with complex-valued signals. Consider the filter

$$y[n] = x[n] + re^{j\theta}y[n-1],$$

where r and θ are both real numbers, $0 < r < 1$, $0 < \theta < \pi$. Obviously, if $x[n]$ is a real-valued signal, $y[n]$ will be complex-valued.

(a) In terms of r and θ, what is the frequency response of this filter?

Solution:

$$H(\omega) = \frac{1}{1 - re^{j\theta}e^{-j\omega}}$$

(b) In terms of r and θ, what is the impulse response of this filter?

Solution:

$$h[n] = r^n e^{j\theta n} u[n]$$

2. (10 points) Suppose you are given two signals, $x[n]$ and $y[n]$. You want to create an M-tap filter, $a[n]$ such that $\hat{y}[n] = a[n] * x[n]$ is a good estimate of $y[n]$, in the sense that it minimizes the sum-squared error:

$$\varepsilon = \sum_{n=-\infty}^{\infty} \left(y[n] - \hat{y}[n]\right)^2$$

where

$$\hat{y}[n] = \sum_{m=0}^{M-1} a[m]x[n-m]$$

Assume that $x[n]$ and $y[n]$ are known; find a set of M linear equations that can be solved to find the M coefficients, $a[0]$ through $a[M-1]$. You don't need to simplify or invent any special notation; just find M equations in M unknowns.

Solution: The M equations are:

$$\sum_{n=-\infty}^{\infty} \left(y[n] - \sum_{m=0}^{M-1} a[m]x[n-m]\right) x[n-k] = 0, \quad 0 \le k \le M-1$$

3. (20 points) Suppose that $x[n_1, n_2]$ is an infinite-sized grayscale gradient image, with the following content:

$$x[n_1, n_2] = 255 - \frac{n_1}{1000}$$

The following three parts specify three different ways in which the image might be filtered. Here $*_1$ denotes convolution in the n_1 direction, $*_2$ denotes convolution in the n_2 direction, and $*$ denotes two-dimensional convolution.

(a)

$$\begin{aligned} h[n] &= 0.5\delta[n+1] - 0.5\delta[n-1] \\ g_1[n_1, n_2] &= h[n_1] *_1 x[n_1, n_2] \end{aligned}$$

Find $g_1[n_1, n_2]$ as a function of n_1 and/or n_2.

Solution:

$$\begin{aligned} g_1[n_1, n_2] &= 0.5x[n_1+1, n_2] - 0.5x[n_1-1, n_2] \\ &= -0.5\frac{n_1+1}{1000} + 0.5\frac{n_1-1}{1000} \\ &= -\frac{1}{1000} \end{aligned}$$

(b)

$$\begin{aligned} h[n] &= 0.5\delta[n+1] - 0.5\delta[n-1] \\ g_2[n_1, n_2] &= h[n_2] *_2 x[n_1, n_2] \end{aligned}$$

Find $g_2[n_1, n_2]$ as a function of n_1 and/or n_2.

Solution:

$$\begin{aligned} g_1[n_1, n_2] &= 0.5x[n_1, n_2+1] - 0.5x[n_1, n_2-1] \\ &= 0 \end{aligned}$$

Problem 3 cont'd

(c)

$$
\begin{aligned}
h[n] &= 0.1\delta[n+2] + 0.3\delta[n+1] + 0.5\delta[n] + 0.3\delta[n-1] + 0.1\delta[n-2] \\
y[n_1, n_2] &= h[n_2] *_2 x[n_1, n_2]
\end{aligned}
$$

Find $y[n_1, n_2]$ as a function of n_1 and/or n_2.

Solution:

$$
\begin{aligned}
y[n_1, n_2] &= \sum_m h[m] x[n_1, n_2 - m] \\
&= \left(255 - \frac{n_1}{1000}\right) \sum_m h[m] \\
&= 1.3 \left(255 - \frac{n_1}{1000}\right)
\end{aligned}
$$

4. (20 points) Suppose you start with a 2×2 image:

$$x[n_1,n_2] = \begin{cases} a & n_1=0, n_2=0, \quad b \quad n_1=0, n_2=1 \\ c & n_1=1, n_2=0, \quad d \quad n_1=1, n_2=1 \\ 0 & \text{otherwise} \end{cases}$$

This image is upsampled by a factor of five, then lowpass filtered, thus

$$y[n_1,n_2] = \begin{cases} x\left[\frac{n_1}{5}, \frac{n_2}{5}\right] & \frac{n_1}{5}, \frac{n_2}{5} \text{ both integers} \\ 0 & \text{otherwise} \end{cases}$$

$$z[n_1,n_2] = h[n_1,n_2] * y[n_1,n_2]$$

In each of the following three cases, find the value of $z[2,4]$ in terms of the constants a, b, c, d.

(a)

$$h[n_1,n_2] = \begin{cases} 1 & 0 \le n_1 < 5, \ 0 \le n_2 < 5 \\ 0 & \text{otherwise} \end{cases}$$

In terms of a, b, c, d, what is $z[2,4]$?

Solution:

$$z[2,4] = a$$

(b)

$$h[n_1,n_2] = \begin{cases} \left(1 - \frac{|n_1|}{5}\right)\left(1 - \frac{|n_2|}{5}\right) & -5 \le n_1 \le 5, \ -5 \le n_2 \le 5 \\ 0 & \text{otherwise} \end{cases}$$

In terms of a, b, c, d, what is $z[2,4]$? Do not simplify explicit numerical expressions.

Solution:

$$z[2,4] = \left(\frac{3}{5}\right)\left(\frac{1}{5}\right)a + \left(\frac{3}{5}\right)\left(\frac{4}{5}\right)b + \left(\frac{2}{5}\right)\left(\frac{1}{5}\right)c + \left(\frac{2}{5}\right)\left(\frac{4}{5}\right)d$$

Problem 4 cont'd

(c)

$$h[n_1, n_2] = \left(\frac{\sin(0.2\pi n_1)}{0.2\pi n_1}\right)\left(\frac{\sin(0.2\pi n_2)}{0.2\pi n_2}\right)$$

You may find it useful to know that

$$\frac{\sin(0.2\pi n)}{0.2\pi n} \approx \begin{cases} 1 & n=0, \quad 0.94 \quad n=1, \\ 0.76 & n=2, \quad 0.50 \quad n=3, \\ 0.23 & n=4, \quad 0.0 \quad\;\; n=5 \end{cases}$$

In terms of a, b, c, d, what is $z[2,4]$? Do not simplify explicit numerical expressions.

Solution:

$$z[2,4] = (0.76)\,(0.23)\,a + (0.76)\,(0.94)\,b + (0.50)\,(0.23)\,c + (0.50)\,(0.94)\,d$$

5. (20 points) Consider the following matrices and vectors:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

In both parts of this problem, the matrix A and the vector $\vec{f}$ are **known**, but the vector $\vec{u}$ is **unknown**. In both parts of this problem, you may assume that A is invertible. In both parts of this problem, your task will be to find equations that can be solved to find the two unknowns u and v in terms of the known quantities A and $\vec{f}$.

(a) For this part of the problem, suppose that f and g are the eigenvalues of A, and are already known. Suppose that $\vec{u}$ is the eigenvector whose eigenvalue is f, and suppose that, therefore, we assume that $u^2 + v^2 = 1$ and $u \geq 0$. Under these assumptions, find an equation that can be solved to find $\vec{u}$. You may express your equation in terms of A, $\vec{f}$, and $\vec{u}$, and/or in terms of any or all of their scalar elements.

Solution:

$$f\vec{u} = A\vec{u}$$

(b) For this part of the problem, none of the assumptions in part (a) are still true. Instead, suppose that $\vec{u}$ has been chosen to minimize the following function:

$$\vec{u}^T A\vec{u} + 3\vec{u}^T \vec{f} + 15$$

Under these assumptions, find an equation that can be solved to find $\vec{u}$. You may express your equation in terms of A, $\vec{f}$, and $\vec{u}$, and/or in terms of any or all of their scalar elements.

Solution:

$$2(A + A^T)\vec{u} + 3\vec{f} = 0$$

6. (10 points) Suppose that you are watching a movie in which the camera is floating to the left at a rate of approximately four columns per frame, so that the optical flow field is uniform everywhere as

$$\vec{v} = \left[\begin{array}{c} v_r \\ v_c \end{array}\right] = \left[\begin{array}{c} 0 \\ -4 \end{array}\right]$$

Suppose that the image $f[r,c]$ is a grayscale gradient, bright at the top right, and dark at the bottom left, i.e.,

$$f[r,c] = 128\left(1 + \frac{c}{640} - \frac{r}{480}\right)$$

What is $\frac{\partial f}{\partial t}$, the change in pixel intensity, as a function of r and c? Note: do not simplify explicit numerical expressions.

Solution: The gradient of the image is

$$\nabla f = \left[\begin{array}{c} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial c} \end{array}\right] = \left[\begin{array}{c} 128/480 \\ -128/640 \end{array}\right]$$

Plugging this into the optical flow equation, we get

$$\begin{aligned} \frac{\partial f}{\partial t} &= -(\nabla f)^T \vec{v} \\ &= 4 \times 128/640 \end{aligned}$$

Signal Processing

$$y[n] = Gx[n] + \sum_{m=1}^{N} a_m y[n-m] = h[n] * x[n]$$

$$H(z) = \frac{1}{1 - \sum_{m=1}^{N} a_m z^{-m}} = \frac{1}{\prod_{k=1}^{N}(1 - p_k z^{-1})} = \sum_{k=1}^{N} \frac{C_k}{1 - p_k z^{-1}}$$

$$h[n] = \sum_{k=1}^{N} C_k p_k^n u[n]$$

Linear Prediction

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} e^2[n] = \sum_{n=-\infty}^{\infty} \left(s[n] - \sum_{m=1}^{p} a_m s[n-m] \right)^2$$

$$0 = \sum_{n=-\infty}^{\infty} \left(s[n] - \sum_{m=1}^{p} a_m s[n-m] \right) s[n-k], \quad 1 \le k \le p$$

$$\vec{\gamma} = R\vec{a}$$

Linear Algebra

If A symmetric, square and nonsingular then

$$A = V\Lambda V^T, \quad \Lambda = V^T A V, \quad VV^T = V^T V = I$$

If A is tall and thin, with full column rank, then

$$A^\dagger \vec{b} = \text{argmin}_{\vec{v}} \|\vec{b} - A\vec{v}\|^2 = (A^T A)^{-1} A^T \vec{b}$$

Image Filtering

$$x[n_1, n_2] * h_1[n_1] h_2[n_2] = h_1[n_1] *_1 (h_2[n_2] *_2 x[n_1, n_2])$$

Image Interpolation

$$y[n_1, n_2] = \begin{cases} x\left[\frac{n_1}{U}, \frac{n_2}{U}\right] & \frac{n_1}{U}, \frac{n_2}{U} \text{ both integers} \\ 0 & \text{otherwise} \end{cases}$$

$$z[n_1, n_2] = h[n_1] *_1 (h[n_2] *_2 y[n_1, n_2])$$

$$h_{\text{rect}}[n] = \begin{cases} 1 & 0 \le n < U \\ 0 & \text{otherwise} \end{cases}, \quad h_{\text{tri}}[n] = \begin{cases} 1 - \frac{|n|}{U} & -U \le n \le U \\ 0 & \text{otherwise} \end{cases}, \quad h_{\text{sinc}}[n] = \frac{sin(\pi n/U)}{\pi n/U}$$

Optical Flow

$$-\frac{\partial f}{\partial t} = (\nabla f)^T \vec{v}$$