UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Spring 2021

PRACTICE EXAM 1

Exam will be Tuesday, September 28, 2021

- This will be a CLOSED BOOK exam.
- You will be permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 9:30am; you will need to photograph and upload your answers by exactly 11:00am.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

1. (16 points) A 200×200 sunset image is bright on the bottom, and dark on top, thus the pixel in the i^{th} row and j^{th} column has intensity $A[i,j] = 200 - i$. Suppose that this formula extends past the edge of the image, and also extends to non-integer pixel locations, so you can assume that it holds for every real-valued coordinate value (i,j).

   Suppose your goal is to recognize sunset images that have been taken by people laying upside-down on the beach. You decide to train the classifier by rotating the image $A[i,j]$ to every possible angle, thus creating the training images

$$B_k[i,j] = A[i\cos\theta_k - \sin\theta_k, i\sin\theta_k + \cos\theta_k], \quad \theta_k = \frac{2\pi k}{100}, \quad 0 \le k \le 99$$

   Your next step is to reshape each 200×200 image $B_k[i,j]$ into a vector of raw pixel intensities, $\vec{x}_k$, then to compute the dataset mean, $\vec{m} = \frac{1}{100}\sum_{k=0}^{99} \vec{x}_k$.

   (a) What is the length of the vector $\vec{m}$?

   Solution:

$$\text{len}(\vec{m}) = \text{len}(\vec{x}_k) = 200 \times 200 = 40,000$$

   (b) What is the numerical value of $\vec{m}$? Provide enough information to specify the value of every element of the vector.

   Solution:

$$\begin{aligned} M[i,j] &= \frac{1}{100}\sum_{k=0}^{99} B_k[i,j] \\ &= \frac{1}{100}\sum_{k=0}^{99} A\left[i\cos\frac{2\pi k}{100} - j\sin\frac{2\pi k}{100}, i\sin\frac{2\pi k}{100} + j\cos\frac{2\pi k}{100}\right] \\ &= \frac{1}{100}\left(200 - i\cos\frac{2\pi k}{100} + j\sin\frac{2\pi k}{100}\right) \\ &= 200 \end{aligned}$$

   Therefore

$$\vec{m} = [200, 200, \ldots, 200]^T$$

2. (16 points) Suppose you have a 1000-sample audio waveform, $x[n]$, such that $x[n] \neq 0$ for $0 \leq n \leq 999$. You want to chop this waveform into 200-sample frames, with 10% overlap between frames. How many nonzero samples are there in the last frame?

Solution: 10% overlap = 20-sample overlap, so the frames start at samples 0, 180, 360, 540, 720, and 900. The last frame has 100 nonzero samples.

3. (21 points) You are given a 640x480 B/W input image, $x[n_1, n_2]$ for integer pixel values $0 \leq n_1 \leq 639$, $0 \leq n_2 \leq 479$. You wish to interpolate the given pixel values in order to find the value of the image at location $(500.3, 300.8)$. Specify the formula used to calculate $x[500.3, 300.8]$ using each of the following algorithms. Be certain that your formula clearly states which pixels from the input image are used.

   (a) Piece-wise constant interpolation.

   Solution:
   $$x[500.3, 300.8] = x[500, 300]$$

   (b) Bilinear interpolation.

   Solution:
   $$\begin{aligned} x[500.3, 300.8] &= (0.7)(0.2)x[500, 300] + (0.7)(0.8)x[500, 301] + \\ &\quad (0.3)(0.2)x[501, 300] + (0.3)(0.8)x[501, 301] \end{aligned}$$

   (c) Sinc interpolation.

   Solution:
   $$x[500.3, 300.8] = \sum_{n_1=0}^{639} \sum_{n_2=0}^{479} x[n_1, n_2] \text{sinc}\left(\pi(500.3 - n_1)\right) \text{sinc}\left(\pi(300.8 - n_2)\right)$$

4. (10 points) Suppose a particular image has the following pixel values:

$$a[0,0] = 1, \quad a[1,0] = 0, \quad a[0,1] = 0, \quad a[1,1] = 0$$

Use bilinear interpolation to estimate the value of the pixel $a\left(\frac{1}{3}, \frac{1}{3}\right)$.

Solution: $a\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{4}{9}$

5. (20 points) Image warping has moved input pixel $i(4.6, 8.2)$ to output pixel $i'(15, 7)$. Input pixel $i(4.6, 8.2)$ is unknown, but you know that $i(4,8) = a$, $i(4,9) = b$, $i(5,8) = c$, and $i(5,9) = d$. Use bilinear interpolation to estimate $i(4.6, 8.2)$ in terms of a, b, c, and d.

Solution:

$$i(4.6, 8.2) = (0.4)(0.8)a + (0.4)(0.2)b + (0.6)(0.8)c + (0.6)(0.2)d$$

6. (17 points) Your goal is to find a positive real number, a, so that $ax[n]$ is as similar as possible to $y[n]$ in the sense that it minimizes the following error:

$$\epsilon = \int_{-\pi}^{\pi} \left(|Y(e^{j\omega})| - a|X(e^{j\omega})|\right)^2 d\omega$$

Find the value of a that mininimizes ϵ, in terms of $|X(e^{j\omega})|$ and $|Y(e^{j\omega})|$.

Solution:

$$\frac{\partial \epsilon}{\partial a} = 2\int_{-\pi}^{\pi} \left(a|X(e^{j\omega})| - |Y(e^{j\omega})|\right) |X(e^{j\omega})| d\omega$$

which equals 0 at:

$$a = \frac{\int_{-\pi}^{\pi} |X(e^{j\omega})||Y(e^{j\omega})| d\omega}{\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega}$$

7. (5 points) Consider the problem of upsampling, by a factor of 2, the infinite-sized image

$$x[n_1,n_2] = \delta[n_1-5] = \begin{cases} 1 & n_1 = 5 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that the image is upsampled, then filtered, as

$$y[n_1,n_2] = \begin{cases} x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\ 0 & \text{otherwise} \end{cases} \qquad z[n_1,n_2] = y[n_1,n_2] ** h[n_1,n_2]$$

Let $h[n_1,n_2]$ be the ideal anti-aliasing filter with frequency response

$$H(\omega_1,\omega_2) = \begin{cases} 1 & |\omega_1| < \frac{\pi}{2}, \quad |\omega_2| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1,n_2]$.

Solution:

$$h[n_1,n_2] = h_1[n_1]h_2[n_2] = \left(\frac{1}{2}\right)\text{sinc}\left(\frac{\pi n_1}{2}\right)\left(\frac{1}{2}\right)\text{sinc}\left(\frac{\pi n_2}{2}\right)$$

$$y[n_1,n_2] = \begin{cases} 1 & n_1 = 10 \text{ and } n_2 \text{ a multiple of } 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \left(\sum_{p=-\infty}^{\infty} \delta[n_2-2p]\right) & n_1 = 10 \\ 0 & \text{otherwise} \end{cases}$$

Convolving along each row gives $h_2[n_2] * y[n_1,n_2]$, which is zero, except on the $n_1 = 10$ row. On that row, $y[n_1,n_2]$ is equal to one on the even-numbered samples, and equal to zero on the odd-numbered samples. The correct answer is the obvious one: the low-pass filter computes a perfect average between 0 and 1, so each pixel winds up with a value of 1/2. If you want to do a more careful analysis, you could notice that this row is an impulse train with a period of $P = 2$, and therefore it has a DTFT which has impulses of area $2\pi/P = \pi$ at $\omega = 0$ and $\omega = \pi$. The LPF keeps only the $\omega = 0$ impulse, thus:

$$\begin{aligned}
h_2[n_2] * y[n_1,n_2] &= \begin{cases} \left(\sum_{p=-\infty}^{\infty} \delta[n_2-2p]\right) * \left(\frac{1}{2}\text{sinc}\left(\frac{\pi n_2}{2}\right)\right) & n_1 = 10 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathcal{F}^{-1}\left\{\left(\frac{2\pi}{2}\sum_{k=0}^{1}\delta\left(\omega - \frac{2\pi k}{2}\right)\right)\left(\begin{cases} 1 & |\omega_2| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}\right)\right\} & n_1 = 10 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathcal{F}^{-1}\left\{\pi\delta(\omega)\right\} & n_1 = 10 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2} & n_1 = 10 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Convolving along each column, then, gives

$$z[n_1,n_2] = h_1[n_1] * h_2[n_2] * y[n_1,n_2] = \left(\frac{1}{4}\right)\text{sinc}\left(\frac{\pi(n_1-10)}{2}\right)$$

8. (10 points) Consider the signal $x[n] = \beta^n u[n]$, where $u[n]$ is the unit step function.

(a) Find the LPC coefficient, α, that minimizes ε, where

$$\varepsilon = \sum_{n=-\infty}^{\infty} e^2[n], \quad e[n] = x[n] - \alpha x[n-1]$$

Solution:

$$\varepsilon = \sum_{n=-\infty}^{\infty} \left(x[n] - \alpha x[n-1]\right)^2 \tag{1}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\beta^n - \alpha\beta^{n-1}\right)^2 \tag{2}$$

Differentiating w.r.t. α gives

$$\frac{\partial \varepsilon}{\partial \alpha} = -2\sum_{n=1}^{\infty} \beta \left(\beta^n - \alpha\beta^{n-1}\right)$$

which is zero iff $\alpha = \beta$.

(b) Find the signal $e[n]$ that results from your choice of α in part (a).

Solution:

$$e[n] = \beta^n u[n] - \alpha\beta^{n-1} u[n-1] = \beta^n \left(u[n] - u[n-1]\right) = \delta[n]$$

9. (10 points) Consider the LPC synthesis filter $s[n] = e[n] + \alpha s[n-1]$.

   (a) Under what condition on α is the synthesis filter stable?

   Solution: The roots of the polynomial $1 - \alpha z^{-1}$ must be inside the unit circle. That's a first-order polynomial, its only root is $z^{-1} = \alpha$, so we just need $|\alpha| < 1$.

   (b) Assume that the synthesis filter is stable. Suppose that $e[n]$ is the pulse train $e[n] = \sum_{p=-\infty}^{\infty} \delta[n-pP]$. As a function of α, P, and ω, what is the DTFT $S(e^{j\omega})$? You need not simplify, but your answer should contain no integrals or infinite sums.

   Solution: The DTFT of the pulse train is a pulse train,

$$E(e^{j\omega}) = \left(\frac{2\pi}{P}\right) \sum_{k=0}^{P-1} \delta\left(\omega - \frac{2\pi k}{P}\right)$$

   The DTFT of the synthesized signal is

$$S(e^{j\omega}) = H(e^{j\omega})E(e^{j\omega}) = \frac{E(e^{j\omega})}{1 - \alpha e^{-j\omega}}$$

   So

$$S(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \left(\frac{2\pi}{P}\right) \sum_{k=0}^{P-1} \delta\left(\omega - \frac{2\pi k}{P}\right)$$

10. (5 points) Consider the synthesis filter $s[n] = e[n] + bs[n-1] - \left(\frac{b}{2}\right)^2 s[n-2]$. For what values of b is the synthesis filter stable?

Solution: Take the Z transform of the difference equation and re-arrange terms, we get

$$S(z)(1 - bz^{-1} + \left(\frac{b}{2}\right)^2 z^{-2}) = E(z)$$

is stable if the roots of the polynomial have absolute value less than 1. The roots of the polynomial are

$$r_k = \frac{b}{2} \pm \frac{\sqrt{b^2 - 4(b/2)^2}}{2} = \frac{b}{2}$$

So $|r_k| < 1$ iff $|b| < 2$.

11. (5 points) Suppose you have a 200×200-pixel image that is just one white dot at pixel $(45, 25)$, and all the other pixels are black:

$$x[n_1, n_2] = \begin{cases} 255 & n_1 = 45,\ n_2 = 25 \\ 0 & \text{otherwise},\ 0 \le n_1 < 199,\ 0 \le n_2 < 199 \end{cases}$$

This image is upsampled to size 400×400, then filtered, as

$$y[n_1, n_2] = \begin{cases} x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\ 0 & \text{otherwise} \end{cases} \qquad z[n_1, n_2] = y[n_1, n_2] ** h[n_1, n_2]$$

where $h[n_1, n_2]$ is the ideal anti-aliasing filter whose frequency response is

$$H(\omega_1, \omega_2) = \begin{cases} 1 & |\omega_1| < \frac{\pi}{2},\ |\omega_2| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1, n_2]$.

Solution:

$$y[n_1, n_2] = \begin{cases} 255 & n_1 = 90, n_2 = 50 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n_1, n_2] = \frac{1}{2}\text{sinc}\left(\frac{\pi n_1}{2}\right) \frac{1}{2}\text{sinc}\left(\frac{\pi n_2}{2}\right) = h_1[n_1] h_2[n_2]$$

Row convolution gives $v[n_1, n_2] = h_2[n_2] * y[n_1, n_2]$, which is

$$v[n_1, n_2] = \begin{cases} \frac{255}{2}\text{sinc}\left(\frac{\pi(n_2-50)}{2}\right) & n_1 = 90 \\ 0 & \text{otherwise} \end{cases}$$

Column convolution then gives

$$z[n_1, n_2] = \frac{255}{4}\text{sinc}\left(\frac{\pi(n_1 - 90)}{2}\right) \text{sinc}\left(\frac{\pi(n_2 - 50)}{2}\right)$$

12. (10 points) Consider an infinite-sized grayscale image of a diagonal gray line:

$$x[n_1, n_2] = \begin{cases} 105 & n_1 - n_2 = 5 \\ 0 & \text{otherwise} \end{cases}, \quad -\infty < n_1 < \infty, \quad -\infty < n_2 < \infty$$

(a) Suppose we **convolve each row** with a differencing filter:

$$y[n_1, n_2] = x[n_1, n_2] * d_2[n_2], \quad d_2[n_2] = \begin{cases} 1 & n_2 = 0 \\ -1 & n_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $y[n_1, n_2]$.

Solution:

$$\begin{aligned} y[n_1, n_2] &= \sum_{m_2} x[n_1, n_2 - m_2] d_2[m_2] \\ &= x[n_1, n_2] - x[n_1, n_2 - 2] \\ &= \begin{cases} 105 & n_2 = n_1 - 5 \\ -105 & n_2 = n_1 - 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) Suppose, INSTEAD, that we **convolve each row** with an averaging filter

$$z[n_1, n_2] = x[n_1, n_2] * a_2[n_2], \quad a_2[n_2] = \begin{cases} 1 & n_2 \in \{0, 2\} \\ 2 & n_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1, n_2]$.

Solution:

$$\begin{aligned} y[n_1, n_2] &= \sum_{m_2} x[n_1, n_2 - m_2] a_2[m_2] \\ &= x[n_1, n_2] + 2x[n_1, n_2 - 1] + x[n_1, n_2 - 2] \\ &= \begin{cases} 105 & n_2 = n_1 - 5 \text{ or } n_1 - 3 \\ 210 & n_2 = n_1 - 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

13. (15 points) A particular two-pole filter has the impulse response

$$h[n] = e^{-\sigma_1 n} \sin(\omega_1 n) u[n]$$

$H(z)$ can be written as

$$H(z) = \frac{Gz^{-1}}{A(z)}, \quad A(z) = 1 - a_1 z^{-1} - a_2 z^{-2}$$

Find a_1, a_2, and G in terms of σ_1 and ω_1.

Solution:

$$\begin{aligned} G &= e^{-\sigma_1} \sin(\omega_1) \\ a_1 &= 2e^{-\sigma_1} \cos(\omega_1) \\ a_2 &= -e^{-2\sigma_1} \end{aligned}$$

14. (5 points) Suppose that $\mathcal{X}$ is the unit disk, i.e.,

$$\mathcal{X} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1^2 + x_2^2 \leq 1 \right\}$$

Suppose that $\mathcal{Y}$ is defined as:

$$\mathcal{X} = \left\{ \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : \vec{y} = A\vec{x} \leq 1 \right\}$$

where A is defined to be the following matrix:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Notice that the area of $\mathcal{X}$, in the two-dimensional plane, is $|\mathcal{X}| = \pi$. What is the numerical value of $|\mathcal{Y}|$, the area of $\mathcal{Y}$?

Solution:

$$|\mathcal{Y}| = |A|\pi = 3\pi$$

15. (5 points) Suppose that you are trying to allocate money to a set of N different possible investments. Suppose that if you allocate a_k dollars to investment k, it will return $a_k b_k$ dollars in profit. However, in order to invest, you first need to borrow the money, and the cost of borrowing is quadratic in terms of the allocations. Let $\vec{a}$ be your vector of allocations, let $\vec{b}$ be the vector of profit factors, and let C be the matrix of cost factors; suppose that your total profit is

$$P = \vec{b}^T \vec{a} - \vec{a}^T C \vec{a}$$

In terms of $\vec{b}$ and C, find the vector $\vec{a}$ that will maximize your profit. You may assume that C is nonsingular.

> **Solution:**
>
> $$\nabla_{\vec{a}} P = \vec{b} - 2C\vec{a}$$
>
> Setting $\nabla_{\vec{a}} P = 0$ gives
>
> $$\vec{a} = \frac{1}{2} C^{-1} \vec{b}$$

16. (10 points) Suppose that you are watching a movie in which the camera is rotating around the top left corner of the frame at a rate of about 0.03 radians/frame, so that, as a function of the row index r and column index c, the optical flow field is

$$\vec{v} = \begin{bmatrix} v_r \\ v_c \end{bmatrix} = \begin{bmatrix} 0.03c \\ -0.03r \end{bmatrix}$$

Suppose that the image $f[r, c]$ is a color gradient, with bright colors at the top of the image, and darker colors at the bottom:

$$f[r, c] = 255 - 0.1r$$

What is $\frac{\partial f}{\partial t}$, the change in pixel intensity, as a function of r and c?

> **Solution:** The gradient of the image is
>
> $$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial c} \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}$$
>
> Plugging this into the optical flow equation, we get
>
> $$\begin{aligned} \frac{\partial f}{\partial t} &= -(\nabla f)^T \vec{v} \\ &= -[-0.1, 0] \begin{bmatrix} 0.03c \\ -0.03r \end{bmatrix} \\ &= 0.003c \end{aligned}$$