UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Spring 2021

PRACTICE EXAM 1

Exam will be Tuesday, September 28, 2021

- This will be a CLOSED BOOK exam.
- You will be permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 9:30am; you will need to photograph and upload your answers by exactly 11:00am.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

1. (16 points) A 200×200 sunset image is bright on the bottom, and dark on top, thus the pixel in the i^{th} row and j^{th} column has intensity $A[i,j] = 200 - i$. Suppose that this formula extends past the edge of the image, and also extends to non-integer pixel locations, so you can assume that it holds for every real-valued coordinate value (i,j).

   Suppose your goal is to recognize sunset images that have been taken by people laying upside-down on the beach. You decide to train the classifier by rotating the image $A[i,j]$ to every possible angle, thus creating the training images

   $$B_k[i,j] = A[i\cos\theta_k - \sin\theta_k, i\sin\theta_k + \cos\theta_k], \quad \theta_k = \frac{2\pi k}{100}, \quad 0 \le k \le 99$$

   Your next step is to reshape each 200×200 image $B_k[i,j]$ into a vector of raw pixel intensities, $\vec{x}_k$, then to compute the dataset mean, $\vec{m} = \frac{1}{100}\sum_{k=0}^{99} \vec{x}_k$.

   (a) What is the length of the vector $\vec{m}$?

   (b) What is the numerical value of $\vec{m}$? Provide enough information to specify the value of every element of the vector.

2. (16 points) Suppose you have a 1000-sample audio waveform, $x[n]$, such that $x[n] \neq 0$ for $0 \leq n \leq 999$. You want to chop this waveform into 200-sample frames, with 10% overlap between frames. How many nonzero samples are there in the last frame?

3. (21 points) You are given a 640x480 B/W input image, $x[n_1, n_2]$ for integer pixel values $0 \leq n_1 \leq 639$, $0 \leq n_2 \leq 479$. You wish to interpolate the given pixel values in order to find the value of the image at location $(500.3, 300.8)$. Specify the formula used to calculate $x[500.3, 300.8]$ using each of the following algorithms. Be certain that your formula clearly states which pixels from the input image are used.

   (a) Piece-wise constant interpolation.

   (b) Bilinear interpolation.

   (c) Sinc interpolation.

4. (10 points) Suppose a particular image has the following pixel values:

$$a[0,0]=1, \quad a[1,0]=0, \quad a[0,1]=0, \quad a[1,1]=0$$

Use bilinear interpolation to estimate the value of the pixel $a\left(\frac{1}{3},\frac{1}{3}\right)$.

5. (20 points) Image warping has moved input pixel $i(4.6,8.2)$ to output pixel $i'(15,7)$. Input pixel $i(4.6,8.2)$ is unknown, but you know that $i(4,8)=a$, $i(4,9)=b$, $i(5,8)=c$, and $i(5,9)=d$. Use bilinear interpolation to estimate $i(4.6,8.2)$ in terms of a,b,c, and d.

6. (17 points) Your goal is to find a positive real number, a, so that $ax[n]$ is as similar as possible to $y[n]$ in the sense that it minimizes the following error:

$$\epsilon = \int_{-\pi}^{\pi} \left(|Y(e^{j\omega})| - a|X(e^{j\omega})|\right)^2 d\omega$$

Find the value of a that mininimizes ϵ, in terms of $|X(e^{j\omega})|$ and $|Y(e^{j\omega})|$.

7. (5 points) Consider the problem of upsampling, by a factor of 2, the infinite-sized image

$$x[n_1, n_2] = \delta[n_1 - 5] = \begin{cases} 1 & n_1 = 5 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that the image is upsampled, then filtered, as

$$y[n_1, n_2] = \begin{cases} x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\ 0 & \text{otherwise} \end{cases} \qquad z[n_1, n_2] = y[n_1, n_2] * *h[n_1, n_2]$$

Let $h[n_1, n_2]$ be the ideal anti-aliasing filter with frequency response

$$H(\omega_1, \omega_2) = \begin{cases} 1 & |\omega_1| < \frac{\pi}{2}, \quad |\omega_2| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1, n_2]$.

8. (10 points) Consider the signal $x[n] = \beta^n u[n]$, where $u[n]$ is the unit step function.

(a) Find the LPC coefficient, α, that minimizes ε, where

$$\varepsilon = \sum_{n=-\infty}^{\infty} e^2[n], \quad e[n] = x[n] - \alpha x[n-1]$$

(b) Find the signal $e[n]$ that results from your choice of α in part (a).

9. (10 points) Consider the LPC synthesis filter $s[n] = e[n] + \alpha s[n-1]$.

   (a) Under what condition on α is the synthesis filter stable?

   (b) Assume that the synthesis filter is stable. Suppose that $e[n]$ is the pulse train $e[n] = \sum_{p=-\infty}^{\infty} \delta[n-pP]$. As a function of α, P, and ω, what is the DTFT $S(e^{j\omega})$? You need not simplify, but your answer should contain no integrals or infinite sums.

10. (5 points) Consider the synthesis filter $s[n] = e[n] + bs[n-1] - \left(\frac{b}{2}\right)^2 s[n-2]$. For what values of b is the synthesis filter stable?

11. (5 points) Suppose you have a 200×200-pixel image that is just one white dot at pixel $(45, 25)$, and all the other pixels are black:

$$x[n_1, n_2] = \begin{cases} 255 & n_1 = 45,\ n_2 = 25 \\ 0 & \text{otherwise},\ 0 \le n_1 < 199,\ 0 \le n_2 < 199 \end{cases}$$

This image is upsampled to size 400×400, then filtered, as

$$y[n_1, n_2] = \begin{cases} x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\ 0 & \text{otherwise} \end{cases} \qquad z[n_1, n_2] = y[n_1, n_2] * *h[n_1, n_2]$$

where $h[n_1, n_2]$ is the ideal anti-aliasing filter whose frequency response is

$$H(\omega_1, \omega_2) = \begin{cases} 1 & |\omega_1| < \frac{\pi}{2},\ |\omega_2| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1, n_2]$.

12. (10 points) Consider an infinite-sized grayscale image of a diagonal gray line:

$$x[n_1, n_2] = \begin{cases} 105 & n_1 - n_2 = 5 \\ 0 & \text{otherwise} \end{cases}, \quad -\infty < n_1 < \infty, \quad -\infty < n_2 < \infty$$

(a) Suppose we **<u>convolve each row</u>** with a differencing filter:

$$y[n_1, n_2] = x[n_1, n_2] * d_2[n_2], \quad d_2[n_2] = \begin{cases} 1 & n_2 = 0 \\ -1 & n_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $y[n_1, n_2]$.

(b) Suppose, INSTEAD, that we **<u>convolve each row</u>** with an averaging filter

$$z[n_1, n_2] = x[n_1, n_2] * a_2[n_2], \quad a_2[n_2] = \begin{cases} 1 & n_2 \in \{0, 2\} \\ 2 & n_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $z[n_1, n_2]$.

13. (15 points) A particular two-pole filter has the impulse response

$$h[n] = e^{-\sigma_1 n} \sin(\omega_1 n) u[n]$$

$H(z)$ can be written as

$$H(z) = \frac{Gz^{-1}}{A(z)}, \quad A(z) = 1 - a_1 z^{-1} - a_2 z^{-2}$$

Find a_1, a_2, and G in terms of σ_1 and ω_1.

14. (5 points) Suppose that $\mathcal{X}$ is the unit disk, i.e.,

$$\mathcal{X} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1^2 + x_2^2 \leq 1 \right\}$$

Suppose that $\mathcal{Y}$ is defined as:

$$\mathcal{X} = \left\{ \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : \vec{y} = A\vec{x} \leq 1 \right\}$$

where A is defined to be the following matrix:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Notice that the area of $\mathcal{X}$, in the two-dimensional plane, is $|\mathcal{X}| = \pi$. What is the numerical value of $|\mathcal{Y}|$, the area of $\mathcal{Y}$?

15. (5 points) Suppose that you are trying to allocate money to a set of N different possible investments. Suppose that if you allocate a_k dollars to investment k, it will return $a_k b_k$ dollars in profit. However, in order to invest, you first need to borrow the money, and the cost of borrowing is quadratic in terms of the allocations. Let $\vec{a}$ be your vector of allocations, let $\vec{b}$ be the vector of profit factors, and let C be the matrix of cost factors; suppose that your total profit is

$$P = \vec{b}^T \vec{a} - \vec{a}^T C \vec{a}$$

In terms of $\vec{b}$ and C, find the vector $\vec{a}$ that will maximize your profit. You may assume that C is nonsingular.

16. (10 points) Suppose that you are watching a movie in which the camera is rotating around the top left corner of the frame at a rate of about 0.03 radians/frame, so that, as a function of the row index r and column index c, the optical flow field is

$$\vec{v} = \begin{bmatrix} v_r \\ v_c \end{bmatrix} = \begin{bmatrix} 0.03c \\ -0.03r \end{bmatrix}$$

Suppose that the image $f[r, c]$ is a color gradient, with bright colors at the top of the image, and darker colors at the bottom:

$$f[r, c] = 255 - 0.1r$$

What is $\frac{\partial f}{\partial t}$, the change in pixel intensity, as a function of r and c?