Lecture 12: Hidden Markov Models

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Outline

Review: Bayesian probabilities and Neural nets

Here's how we can estimate the four Bayesian probabilities using a neural net:

1. **Prior:**

$$p(q=i) = \frac{\#\text{ times } q=i \text{ occurred in training data}}{\#\text{ frames in training data}}$$

2. **Posterior:**

$$p(q=i|\vec{x}) = \text{softmax}\left(e[i]\right)$$

Review: Bayesian probabilities and Neural nets

Here's how we can estimate the four Bayesian probabilities using a neural net:

1. **Evidence:**

$$p(\vec{x}) = G \sum_j \exp(e[j])$$

   for some unknown value of G.

2. **Likelihood:**

$$p(\vec{x}|q = i) = \frac{G \exp(e[i])}{p(q = i)}$$

We have to be a little careful in our derivations, but usually we can just choose some value of G with good numerical properties, like $G = 1/\max_j \exp(e[j])$ for example.

Outline

Notation: Inputs and Outputs

- Let's assume we have T consecutive observations, $X = [\vec{x}_1, \ldots, \vec{x}_T]$.
- A "hidden Markov model" represents those probabilities by assuming some sort of "hidden" state sequence, $Q = [q_1, \ldots, q_T]$, where q_t is the hidden (unknown) state variable at time t.

The idea is, can we model these probabilities well enough to solve problems like:

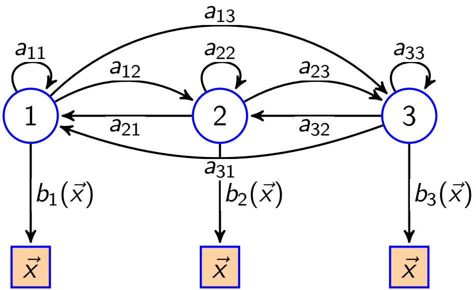
1. **Recognition:** What's $p(X)$ given the model?
2. **Segmentation:** What state is the model in at time t?
3. **Training:** Can we learn a model to fit some data?

HMM: Key Concepts

An HMM is a "generative model," meaning that it models the joint probability $p(Q, X)$ using a model of the way in which those data might have been generated. An HMM pretends the following generative process:

1. Start in state $q_t = i$ with pmf $\pi_i = p(q_1 = i)$.
2. Generate an observation, $\vec{x}$, with pdf $b_i(\vec{x}) = p(\vec{x}|q_t = i)$.
3. Transition to a new state, $q_{t+1} = j$, according to pmf $a_{ij} = p(q_{t+1} = j|q_t = i)$.
4. Repeat.

HMM: Finite State Diagram



1. Start in state $q_t = i$, for some $1 \leq i \leq N$.
2. Generate an observation, $\vec{x}$, with pdf $b_i(\vec{x})$.
3. Transition to a new state, $q_{t+1} = j$, according to pmf a_{ij}.
4. Repeat steps #2 and #3, T times each.

Notation: Model Parameters

Solving an HMM is possible if you **carefully keep track of notation**. Here's standard notation for the parameters:

- $\pi_i = p(q_1 = i)$ is called the **initial state probability**. Let N be the number of different states, so that $1 \le i \le N$.
- $a_{ij} = p(q_t = j | q_{t-1} = i)$ is called the **transition probability**, $1 \le i, j \le N$.
- $b_j(\vec{x}) = p(\vec{x}_t = \vec{x} | q_t = j)$ is called the **observation probability**. It is usually estimated by a neural network, though simpler models (e.g., Gaussians, lookup tables) are possible.
- Λ is the complete set of **model parameters**, including all the π_i's and a_{ij}'s, and the neural net parameters necessary to compute $b_j(\vec{x})$.

The Three Problems for an HMM

1. **Recognition:** Given two different HMMs, Λ_1 and Λ_2, and an observation sequence X. Which HMM was more likely to have produced X? In other words, $p(X|\Lambda_1) > p(X|\Lambda_2)$?
2. **Segmentation:** What is $p(q_t = i|X, \Lambda)$?
3. **Training:** Given an initial HMM Λ, and an observation sequence X, can we find Λ' such that $p(X|\Lambda') > p(X|\Lambda)$?

Outline

The HMM Recognition Problem

- Given
  - $X = [\vec{x}_1, \ldots, \vec{x}_T]$ and
  - $\Lambda = \{\pi_i, a_{ij}, b_j(\vec{x}) \forall i, j\}$,

  what is $p(X|\Lambda)$?
- Let's solve a simpler problem first:
- Given
  - $X = [\vec{x}_1, \ldots, \vec{x}_T]$ and
  - $Q = [q_1, \ldots, q_T]$ and
  - $\Lambda = \{\pi_i, a_{ij}, b_j(\vec{x}) \forall i, j\}$,

  what is $p(X|\Lambda)$?

Joint Probability of State Sequence and Observation Sequence

The joint probability of the state sequence and the observation sequence is calculated iteratively, from beginning to end:

- The probability that $q_1 = q_1$ is π_{q_1}.
- Given q_1, the probability of $\vec{x}_1$ is $b_{q_1}(\vec{x}_1)$.
- Given q_1, the probability of q_2 is $a_{q_1 q_2}$.
- . . . and so on. . .

$$p(Q, X|\Lambda) = \pi_{q_1} b_{q_1}(\vec{x}_1) \prod_{t=2}^{T} a_{q_{t-1} q_t} b_{q_t}(\vec{x}_t)$$

Probability of the Observation Sequence

The probability of the observation sequence, alone, is somewhat harder, because we have to solve this sum:

$$\begin{aligned} p(X|\Lambda) &= \sum_{Q} p(Q, X|\Lambda) \\ &= \sum_{q_T=1}^{N} \cdots \sum_{q_1=1}^{N} p(Q, X|\Lambda) \end{aligned}$$

On the face of it, this calculation seems to have complexity $\mathcal{O}\left\{N^T\right\}$. So for a very small 100-frame utterance, with only 10 states, we have a complexity of $\mathcal{O}\left\{10^{100}\right\}$ =one google.

The Forward Algorithm

The solution is to use a kind of dynamic programming algorithm, called "the forward algorithm." The forward probability is defined as follows:

$$\alpha_t(i) \equiv p(\vec{x}_1, \ldots, \vec{x}_t, q_t = i|\Lambda)$$

Obviously, if we can find $\alpha_t(i)$ for all i and all t, we will have solved the recognition problem, because

$$\begin{aligned} p(X|\Lambda) &= p(\vec{x}_1, \ldots, \vec{x}_T|\Lambda) \\ &= \sum_{i=1}^{N} p(\vec{x}_1, \ldots, \vec{x}_T, q_T = i|\Lambda) \\ &= \sum_{i=1}^{N} \alpha_T(i) \end{aligned}$$

The Forward Algorithm

So, working with the definition $\alpha_t(i) \equiv p(\vec{x}_1, \ldots, \vec{x}_t, q_t = i|\Lambda)$, let's see how we can actually calculate $\alpha_t(i)$.

1. **Initialize:**

$$\begin{aligned}\alpha_1(i) &= p(q_1 = i, \vec{x}_1|\Lambda) \\ &= p(q_1 = i|\Lambda)p(\vec{x}_1|q_1 = i, \Lambda) \\ &= \pi_i b_i(\vec{x}_1)\end{aligned}$$

The Forward Algorithm

Definition: $\alpha_t(i) \equiv p(\vec{x}_1, \ldots, \vec{x}_t, q_t = i|\Lambda)$.

1. **Initialize:**

$$\alpha_1(i) = \pi_i b_i(\vec{x}_1), \quad 1 \leq i \leq N$$

2. **Iterate:**

$$\begin{aligned}
\alpha_t(j) &= p(\vec{x}_1, \ldots, \vec{x}_t, q_t = j|\Lambda) \\
&= \sum_{i=1}^{N} p(\vec{x}_1, \ldots, \vec{x}_{t-1}, q_{t-1} = i) p(q_t = j|q_{t-1} = i) p(\vec{x}_t|q_t = j) \\
&= \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(\vec{x}_t)
\end{aligned}$$

The Forward Algorithm

So, working with the definition $\alpha_t(i) \equiv p(\vec{x}_1, \ldots, \vec{x}_t, q_t = i|\Lambda)$, let's see how we can actually calculate $\alpha_t(i)$.

1. **Initialize:**
$$\alpha_1(i) = \pi_i b_i(\vec{x}_1), \quad 1 \leq i \leq N$$

2. **Iterate:**
$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(\vec{x}_t), \;\; 1 \leq j \leq N, \; 2 \leq t \leq T$$

3. **Terminate:**
$$p(X|\Lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

The Forward Algorithm: Computational Complexity

Most of the computational complexity is in this step:

- **Iterate:**

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(\vec{x}_t), \quad 1 \le i,j \le N, \ 2 \le t \le T$$

Its complexity is:

- For each of $T-1$ time steps, $2 \le t \le T$,...
- we need to calculate N different alpha-variables, $\alpha_t(j)$, for $1 \le j \le N$,...
- each of which requires a summation with N terms.

So the total complexity is $\mathcal{O}\left\{TN^2\right\}$. For example, with $N = 10$ and $T = 100$, the complexity is only 10,000 multiplies.

Outline

The Segmentation Problem

There are different ways to define the segmentation problem. Let's define it this way:

- We want to find the most likely state, $q_t = i$, at time $t, \ldots$
- given knowledge of the *entire* sequence $X = [\vec{x}_1, \ldots, \vec{x}_T]$, not just the current observation. So for example, we don't want to recognize state i at time t if the surrounding observations, $\vec{x}_{t-1}$ and $\vec{x}_{t+1}$, make it obvious that this choice is impossible. Also,...
- given knowledge of the HMM that produced this sequence, Λ.

In other words, we want to find the **state posterior probability**, $p(q_t = i | X, \Lambda)$. Let's define some more notation for the state posterior probability, let's call it

$$\gamma_t(i) = p(q_t = i | X, \Lambda)$$

Use Bayes' Rule

Suppose we already knew the **joint probability,** $p(X, q_t = i|\Lambda)$. Then we could find the state posterior using Bayes' rule:

$$\gamma_t(i) = p(q_t = i|X, \Lambda) = \frac{p(X, q_t = i|\Lambda)}{\sum_{j=1}^N p(X, q_t = j|\Lambda)}$$

Use the Forward Algorithm

Let's expand this:

$$p(X, q_t = i|\Lambda) = p(q_t = i, \vec{x}_1, \ldots, \vec{x}_T|\Lambda)$$

We already know about half of that:
$\alpha_t(i) = p(q_t = i, \vec{x}_1, \ldots, \vec{x}_t|\Lambda)$. We're only missing this part:

$$p(X, q_t = i|\Lambda) = \alpha_t(i) p(\vec{x}_{t+1}, \ldots, \vec{x}_T|q_t = i, \Lambda)$$

Again, let's try the trick of "solve the problem by inventing new notation." Let's define

$$\beta_t(i) \equiv p(\vec{x}_{t+1}, \ldots, \vec{x}_T|q_t = i, \Lambda)$$

The Backward Algorithm

Now let's use the definition $\beta_t(i) \equiv p(\vec{x}_{t+1}, \ldots, \vec{x}_T | q_t = i, \Lambda)$, and see how we can compute that.

1. **Initialize:**

$$\beta_T(i) = 1, \quad 1 \leq i \leq N$$

This might not seem immediately obvious, but think about it. Given that there are no more $\vec{x}$ vectors after time T, what is the probability that there are no more $\vec{x}$ vectors after time T? Well, 1, obviously.

The Backward Algorithm

Now let's use the definition $\beta_t(i) \equiv p(\vec{x}_{t+1}, \ldots, \vec{x}_T | q_t = i, \Lambda)$, and see how we can compute that.

1. **Initialize:**

$$\beta_T(i) = 1, \quad 1 \leq i \leq N$$

2. **Iterate:**

$$\begin{aligned}
\beta_t(i) &= p(\vec{x}_{t+1}, \ldots, \vec{x}_T | q_t = i, \Lambda) \\
&= \sum_{j=1}^{N} p(q_{t+1} = j | q_t = i) p(\vec{x}_{t+1} | q_{t+1} = j) p(\vec{x}_{t+2}, \ldots, \vec{x}_T | q_{t+1} = j) \\
&= \sum_{j=1}^{N} a_{ij} b_j(\vec{x}_{t+1}) \beta_{t+1}(j)
\end{aligned}$$

The Backward Algorithm

Now let's use the definition $\beta_t(i) \equiv p(\vec{x}_{t+1}, \ldots, \vec{x}_T | q_t = i, \Lambda)$, and see how we can compute that.

1. **Initialize:**
$$\beta_T(i) = 1, \quad 1 \le i \le N$$
2. **Iterate:**
$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(\vec{x}_{t+1}) \beta_{t+1}(j), \quad 1 \le i \le N,\ 1 \le t \le T-1$$
3. **Terminate:**
$$p(X|\Lambda) = \sum_{i=1}^{N} \pi_i b_i(\vec{x}_1) \beta_1(i)$$

The Backward Algorithm: Computational Complexity

Most of the computational complexity is in this step:

- **Iterate:**

$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(\vec{x}_{t+1}) \beta_{t+1}(j), \quad 1 \le i \le N, \; 2 \le t \le T$$

Its complexity is:

- For each of $T-1$ time steps, $1 \le t \le T-1, \ldots$
- we need to calculate N different beta-variables, $\beta_t(i)$, for $1 \le i \le N, \ldots$
- each of which requires a summation with N terms.

So the total complexity is $\mathcal{O}\left\{TN^2\right\}$.

Use Bayes' Rule

The segmentation probability is then

$$\begin{aligned}\gamma_t(i) &= \frac{p(X, q_t = i|\Lambda)}{\sum_{k=1}^{N} p(X, q_t = k|\Lambda)} \\ &= \frac{p(\vec{x}_1, \ldots, \vec{x}_t, q_t = i|\Lambda)p(\vec{x}_{t+1}, \ldots, \vec{x}_T|q_t = i, \Lambda)}{\sum_{k=1}^{N} p(\vec{x}_1, \ldots, \vec{x}_t, q_t = k|\Lambda)p(\vec{x}_{t+1}, \ldots, \vec{x}_T|q_t = k, \Lambda)} \\ &= \frac{\alpha_t(i)\beta_t(i)}{\sum_{k=1}^{N} \alpha_t(k)\beta_t(k)}\end{aligned}$$

What About State Sequences?

- Notice a problem: $\gamma_t(i)$ only tells us about one frame at a time! It doesn't tell us anything about the probability of a sequence of states, covering a sequence of frames!
- ...but we can extend the same reasoning to cover two or more consecutive frames. For example, let's define:

$$\xi_t(i,j) = p(q_t = i, q_{t+1} = j | X, \Lambda)$$

- We can solve for $\xi_t(i,j)$ using the same reasoning that we used for $\gamma_t(i)$!

Segmentation: The Backward Algorithm

In summary, we now have three new probabilities, all of which can be computed in $\mathcal{O}\left\{TN^2\right\}$ time:

1. **The Backward Probability:**

$$\beta_t(i) = p(\vec{x}_{t+1}, \ldots, \vec{x}_T | q_t = i, \Lambda)$$

2. **The State Posterior:**

$$\gamma_t(i) = p(q_t = i | X, \Lambda) = \frac{\alpha_t(i)\beta_t(i)}{\sum_{k=1}^N \alpha_t(k)\beta_t(k)}$$

3. **The Segment Posterior:**

$$\begin{aligned}\xi_t(i,j) &= p(q_t = i, q_{t+1} = j | X, \Lambda)\\ &= \frac{\alpha_t(i) a_{ij} b_j(\vec{x}_{t+1})\beta_{t+1}(j)}{\sum_{k=1}^N \sum_{\ell=1}^N \alpha_t(k) a_{k\ell} b_\ell(\vec{x}_{t+1})\beta_{t+1}(\ell)}\end{aligned}$$

Outline

Maximum Likelihood Training

Suppose we're given several observation sequences of the form $X = [\vec{x}_1, \ldots, \vec{x}_T]$. Suppose, also, that we have some initial guess about the values of the model parameters (our initial guess doesn't have to be very good). Maximum likelihood training means we want to compute a new set of parameters, $\Lambda' = \left\{\pi'_i, a'_{ij}, b'_j(\vec{x})\right\}$ that maximize $p(X|\Lambda')$.

1. **Initial State Probabilities:** Find values of π'_i, $1 \le i \le N$, that maximize $p(X|\Lambda')$.
2. **Transition Probabilities:** Find values of a'_{ij}, $1 \le i, j \le N$, that maximize $p(X|\Lambda')$.
3. **Observation Probabilities:** Find values of the neural network parameters such that $b'_j(\vec{x})$ maximizes $p(X|\Lambda')$.

Maximum Likelihood Training with Known State Sequence

Impossible assumption: Suppose that we actually know the state sequences, $Q = [q_1, \ldots, q_T]$, matching with each observation sequence $X = [\vec{x}_1, \ldots, \vec{x}_T]$. Then the maximum likelihood parameters (the Λ' that maximizes $p(X, Q|\Lambda')$) would be given by

1. **Initial State Probabilities:**

$$\pi_i' = \frac{\#\text{ state sequences that start with } q_1 = i}{\#\text{ state sequences in training data}}$$

2. **Transition Probabilities:**

$$a_{ij}' = \frac{\#\text{ frames in which } q_{t-1} = i, q_t = j}{\#\text{ frames in which } q_{t-1} = i}$$

3. **Observation Probabilities:** Re-estimate the neural network in order to maximize the log likelihood of the actual state sequence, i.e., minimize the following loss function:

$$\mathcal{L} = -\sum_{t=1}^{T} \ln b_{q_t}(\vec{x}_t)$$

Expectation Maximization

When the true state sequence is unknown, then we can't maximize the likelihood $p(X, Q|\Lambda')$ directly. Instead, we maximize the *expected* log likelihood, $E_Q\left[\ln p(X, Q|\Lambda')\right]$, where the expectation is over the unknown (hidden) state sequence.

1. **Initial State Probabilities:**

$$\pi_i' = \frac{E\left[\#\text{ state sequences that start with } q_1 = i\right]}{\#\text{ state sequences in training data}}$$

2. **Transition Probabilities:**

$$\pi_i' = \frac{E\left[\#\text{ frames in which } q_{t-1} = i, q_t = j\right]}{E\left[\#\text{ frames in which } q_{t-1} = i\right]}$$

3. **Observation Probabilities:**

$$\mathcal{L} = -\sum_{t=1}^{T} E\left[\ln b_{q_t}(\vec{x}_t)\right]$$

Calculating the Expectations

Now let's talk about how to calculate those expectations.

- In the t^{th} frame, the event $q_t = i, q_{t+1} = j$ either happens, or it doesn't happen.
- So the following expectation is actually just a probability:

$$E\left[\#\text{ times during the } t^{\text{th}} \text{ frame, in which } q_t = i, q_{t+1} = j\right]$$
$$= p(q_t = i, q_{t+1} = j)$$

- Now we need to ask, in order to compute $p(q_t = i, q_{t+1} = j)$, what other information do we get to use? The answer is: the more information you can use, the better your answer will be. So if we already have a previous estimate of Λ, and we know X, then let's use them:

$$E\left[\#\text{ times, during just one frame, in which } q_t = i, q_{t+1} = j\right]$$
$$= p(q_t = i, q_{t+1} = j | X, \Lambda)$$
$$= \xi_t(i,j)$$

The Baum-Welch Algorithm

1. **Initial State Probabilities:**

$$\pi'_i = \frac{E\left[\#\text{ state sequences that start with } q_1 = i\right]}{\#\text{ state sequences in training data}}$$
$$= \frac{\sum_{sequences} \gamma_1(i)}{\#\text{ sequences}}$$

The Baum-Welch Algorithm

1. **Initial State Probabilities:**

$$\pi_i' = \frac{\sum_{sequences} \gamma_1(i)}{\#\ \text{sequences}}$$

2. **Transition Probabilities:**

$$a_{ij}' = \frac{E\left[\#\ \text{frames in which}\ q_{t-1} = i, q_t = j\right]}{E\left[\#\ \text{frames in which}\ q_{t-1} = i\right]}$$

$$= \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{j=1}^{N} \sum_{t=1}^{T-1} \xi_t(i,j)}$$

The Baum-Welch Algorithm

1. **Initial State Probabilities:**

$$\pi_i' = \frac{\sum_{sequences} \gamma_1(i)}{\#\ \text{sequences}}$$

2. **Transition Probabilities:**

$$a_{ij}' = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{j=1}^{N} \sum_{t=1}^{T-1} \xi_t(i,j)}$$

3. **Observation Probabilities:**

$$\begin{aligned}\mathcal{L} &= -\sum_{t=1}^{T} E\left[\ln b_{q_t}(\vec{x}_t)\right] \\ &= -\sum_{t=1}^{T} \sum_{i=1}^{N} \gamma_t(i) \ln b_i(\vec{x}_t)\end{aligned}$$

The Baum-Welch Algorithm

1. **Initial State Probabilities:**

$$\pi_i' = \frac{\sum_{sequences} \gamma_1(i)}{\#\ \text{sequences}}$$

2. **Transition Probabilities:**

$$a_{ij}' = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{j=1}^{N} \sum_{t=1}^{T-1} \xi_t(i,j)}$$

3. **Observation Probabilities:**

$$\mathcal{L} = -\sum_{t=1}^{T} \sum_{i=1}^{N} \gamma_t(i) \ln b_i(\vec{x}_t)$$

Outline

Example: Gumball Machines

"Gumball machines in a Diner at Dallas, Texas, in 2008," Andreas Praefcke, public domain image.

Example: Gumball Machines

- **Observation Probabilities:** Suppose we have two gumball machines, $q = 1$ and $q = 2$. Machine $\#1$ contains 60% Grapefruit gumballs, 40% Apple gumballs. Machine $\#2$ contains 90% Apple, 10% Grapefruit.

$$b_1(x) = \begin{cases} 0.4 & x = A \\ 0.6 & x = G \end{cases}, \quad b_2(x) = \begin{cases} 0.9 & x = A \\ 0.1 & x = G \end{cases}$$

- **Initial State Probabilities:** My friend George flips a coin to decide which machine to use first.

$$\pi_i = 0.5, \quad i \in \{1, 2\}$$

- **Transition Probabilities:** After he's used a machine, George flips two coins, and he only changes machines if both coins come up heads.

$$a_{ij} = \begin{cases} 0.75 & i = j \\ 0.25 & i \neq j \end{cases}$$

A Segmentation Problem

- George bought three gumballs, using three quarters. The three gumballs are $(x_1 = A, x_2 = G, x_3 = A)$.
- Unfortunately, George is a bit of a goofball. The second of the three "quarters" was actually my 1867 silver "Seated Liberty" dollar, worth $4467.
- Which of the two machines do I need to dismantle in order to get my coin back?

Image used with permission of the National Numismatic Collection, National Museum of American History.

The Forward Algorithm: $t = 1$

Remember, the observation sequence is $X = (A, G, A)$.

$$\begin{aligned}
\alpha_1(i) &= \pi_i b_1(i) \\
&= \begin{cases} (0.5)(0.4) = 0.2 & i = 1 \\ (0.5)(0.9) = 0.45 & i = 2 \end{cases}
\end{aligned}$$

The Forward Algorithm: $t = 2$

Remember, the observation sequence is $X = (A, G, A)$.

$$\alpha_2(j) = \sum_{i=1}^{2} \alpha_1(i) a_{ij} b_j(x_2)$$

$$= \begin{cases} \alpha_1(1)a_{11}b_1(x_2) + \alpha_1(2)a_{21}b_1(x_2) & j = 1 \\ \alpha_1(1)a_{12}b_2(x_2) + \alpha_1(2)a_{22}b_2(x_2) & j = 2 \end{cases}$$

$$= \begin{cases} (0.2)(0.75)(0.6) + (0.45)(0.25)(0.6) = 0.04125 & j = 1 \\ (0.2)(0.25)(0.1) + (0.45)(0.75)(0.1) = 0.03875 & j = 2 \end{cases}$$

The Backward Algorithm: $t = 3$

The backward algorithm always starts out with $\beta_T(i) = 1$!

$$\beta_3(i) = 1, \quad i \in \{1, 2\}$$

The Backward Algorithm: $t = 2$

Remember, the observation sequence is $X = (A, G, A)$.

$$\beta_2(i) = \sum_{j=1}^{2} a_{ij} b_j(x_3) \beta_3(j)$$

$$= \begin{cases} a_{11} b_1(x_3) + a_{12} b_2(x_3) & i = 1 \\ a_{21} b_1(x_3) + a_{22} b_2(x_3) & i = 2 \end{cases}$$

$$= \begin{cases} (0.75)(0.4) + (0.25)(0.9) = 0.525 & j = 1 \\ (0.25)(0.4) + (0.75)(0.9) = 0.775 & j = 2 \end{cases}$$

The Solution to the Puzzle

Given the observation sequence is $X = (A, G, A)$, the posterior state probability is

$$\gamma_2(i) = \frac{\alpha_2(i)\beta_2(i)}{\sum_{k=1}^{2} \alpha_2(k)\beta_2(k)}$$
$$= \begin{cases} \frac{(0.04125)(0.525)}{(0.04125)(0.525)+(0.03875)(0.775)} = 0.42 & i = 1 \\ \frac{(0.03875)(0.775)}{(0.04125)(0.525)+(0.03875)(0.775)} = 0.58 & i = 2 \end{cases}$$

So I shoud dismantle gumball machine #2, hoping to find my rare 1867 silver dollar. Good luck!

Outline

The Forward Algorithm

Definition: $\alpha_t(i) \equiv p(\vec{x}_1, \ldots, \vec{x}_t, q_t = i|\Lambda)$. Computation:

1. **Initialize:**
$$\alpha_1(i) = \pi_i b_i(\vec{x}_1), \quad 1 \le i \le N$$
2. **Iterate:**
$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(\vec{x}_t), \quad 1 \le j \le N,\ 2 \le t \le T$$
3. **Terminate:**
$$p(X|\Lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

The Backward Algorithm

Definition: $\beta_t(i) \equiv p(\vec{x}_{t+1}, \ldots, \vec{x}_T | q_t = i, \Lambda)$. Computation:

1. **Initialize:**
$$\beta_T(i) = 1, \quad 1 \le i \le N$$
2. **Iterate:**
$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(\vec{x}_{t+1}) \beta_{t+1}(j), \quad 1 \le i \le N, \ 1 \le t \le T-1$$
3. **Terminate:**
$$p(X|\Lambda) = \sum_{i=1}^{N} \pi_i b_i(\vec{x}_1) \beta_1(i)$$

The Baum-Welch Algorithm

1. **Initial State Probabilities:**

$$\pi_i' = \frac{\sum_{sequences} \gamma_1(i)}{\#\ \text{sequences}}$$

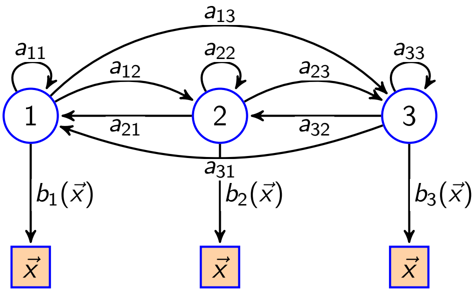
2. **Transition Probabilities:**

$$a_{ij}' = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{j=1}^{N} \sum_{t=1}^{T-1} \xi_t(i,j)}$$

3. **Observation Probabilities:**

$$\mathcal{L} = -\sum_{t=1}^{T} \sum_{i=1}^{N} \gamma_t(i) \ln b_i(\vec{x}_t)$$

Hidden Markov Model



1. Start in state $q_t = i$ with pmf π_i.
2. Generate an observation, $\vec{x}$, with pdf $b_i(\vec{x})$.
3. Transition to a new state, $q_{t+1} = j$, according to pmf a_{ij}.
4. Repeat.