UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Spring 2016

EXAM 2

Thursday, March 31, 2016

- This is a CLOSED BOOK exam. You may use one sheet (front and back) of hand-written notes.
- No calculators are permitted. You need not simplify explicit numerical expressions.
- There are a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score
1	
2	
3	
4	
5	
Total	

Name: ____

Possibly Useful Formulas

Gaussian (Normal) Distribution A Gaussian is parameterized by $\vec{\mu}$, Σ, and $D = \dim(\vec{\mu})$ as

$$\mathcal{N}(\vec{x}|\vec{\mu},\Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

Gaussian Mixture Model (GMM) A GMM is parameterized by c_k, $\vec{\mu}_k$, and Σ_k for $1 \le k \le K$ as

$$p_X(\vec{x}) = \sum_{k=1}^{K} c_k \mathcal{N}(\vec{x}|\vec{\mu}_k,\Sigma_k)$$

Hidden Markov Model (HMM) An HMM is parameterized by $\lambda = \{\pi_i, a_{ij}, b_j(\vec{x})\}$, where

$$\begin{aligned}
\pi_i &= p(q_1 = i|\lambda), \quad 1 \le i \le N \\
a_{ij} &= p(q_{t+1} = j|q_t = i,\lambda), \quad 1 \le i,j \le N \\
b_j(\vec{x}) &= p(\vec{x}|q_t = j,\lambda), \quad 1 \le j \le N
\end{aligned}$$

The acoustic model $b_j(\vec{x})$ might be GMM, for example, in which case the HMM parameters include

$$\begin{aligned}
c_{jk} &= p(g_t = k|q_t = j) \\
\vec{\mu}_{jk} &= E\left[\vec{x}_t|q_t = j, g_t = k\right] \\
\Sigma_{jk} &= E\left[(\vec{x}_t - \vec{\mu}_{jk})(\vec{x}_t - \vec{\mu}_{jk})^T|q_t = j, g_t = k\right]
\end{aligned}$$

Scaled Forward Algorithm

$$\begin{aligned}
\hat{\alpha}_1(i) &= \pi_i b_i(\vec{x}_1), \quad 1 \le i \le N \\
g_1 &= \sum_{i=1}^{N} \hat{\alpha}_1(i) \\
\tilde{\alpha}_1(i) &= \frac{1}{g_1}\hat{\alpha}_1(i) \\
\hat{\alpha}_t(j) &= \sum_{i=1}^{N} \tilde{\alpha}_{t-1}(i) a_{ij} b_j(\vec{x}_t) \\
g_t &= \sum_{j=1}^{N} \hat{\alpha}_t(j) \\
\tilde{\alpha}_t(j) &= \frac{1}{g_t}\hat{\alpha}_t(j) \\
p(\vec{x}_1,\ldots,\vec{x}_t|\lambda) &= \prod_{t=1}^{T} g_t
\end{aligned}$$

Problem 1 (20 points)

You want to classify zoo animals. Your zoo only has two species: elephants and giraffes. There are more elephants than giraffes: if Y is the species,

$$p_Y(\text{elephant}) = \frac{e}{e+1}$$
$$p_Y(\text{giraffe}) = \frac{1}{e+1}$$

where $e = 2.718\ldots$ is the base of the natural logarithm. The height of giraffes is Gaussian, with mean $\mu_G = 5$ meters and variance $\sigma_G^2 = 1$. The height of elephants is also Gaussian, with mean $\mu_E = 3$ and variance $\sigma_E^2 = 1$. Under these circumstances, the minimum probability of error classifier is

$$\hat{y}(x) = \begin{cases} \text{giraffe} & x > \theta \\ \text{elephant} & x < \theta \end{cases}$$

Find the value of θ that minimizes the probability of error.

Problem 2 (20 points)

A 3-nearest neighbor (3NN) estimator of $p_{Y|X}(y_0|\vec{x}_0)$ is computed by finding the 3 nearest neighbors of vector $\vec{x}_0 = [x_{10}, x_{20}]^T$, then measuring

$$p_{Y|X}(y_0|\vec{x}_0) = \frac{\text{\# neighbors from class } y_0}{3}$$

Suppose that the training dataset contains six labeled $(\vec{x}_n, y_n)$ pairs, given by

$$[y_1, y_2, y_3, y_4, y_5, y_6] = [0, 0, 0, 1, 1, 1]$$

$$[\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5, \vec{x}_6] = \left[\begin{array}{cccccc} 0 & 0 & 0 & 2 & 2 & 2 \\ -1 & 0 & 1 & -3 & 0 & 3 \end{array}\right]$$

Find the 3NN estimator $p_{Y|X}(y_0 = 1|\left[\begin{array}{c} x_{10} \\ 0 \end{array}\right])$ as a function of $\vec{x}_0 = \left[\begin{array}{c} x_{10} \\ 0 \end{array}\right]$, that is, for $x_{20} = 0$.

Problem 3 (20 points)

Random vector X is distributed as

$$p_X(\vec{x}) = \sum_{k=1}^{2} c_k \mathcal{N}(\vec{x}|\vec{\mu}, \Sigma)$$

where $c_1 = c_2 = 0.5$, and

$$\vec{\mu}_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad \vec{\mu}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Draw a contour plot showing $p_X(\vec{x})$ as a function of $\vec{x}$. Mark the modes of the distribution, and draw contour lines at levels of $e^{-1/2}$ and e^{-2} times the height of the modes.

Problem 4 (20 points)

A particular hidden Markov model is parameterized by $\lambda = \{\pi_i, a_{ij}, b_j(\vec{x})\}$ where π_i is uniform ($\pi_i = \frac{1}{N}$). Devise an algorithm to compute $p(q_1 = k|\vec{x}_1, \ldots, \vec{x}_T, \lambda)$. Your algorithm should be similar to the forward algorithm, but with a different initialization.

Problem 5 (20 points)

The scaled forward algorithm is provided for you on the formula page at the beginning of this exam. In terms of the quantities $\pi_i, a_{ij}, b_j(\vec{x}), \hat{\alpha}_t(j), g_t$, and/or $\tilde{\alpha}_t(j)$, find a formula for the quantity $p(q_{t-1} = i, q_t = j, \vec{x}_{t-1}, \vec{x}_t | \vec{x}_1, \ldots, \vec{x}_{t-2}, \lambda)$.