UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Principles of Signal Analysis
Spring 2014

**EXAM 2**

Tuesday, April 1, 2014

- This is a CLOSED BOOK exam.
- There are a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| Total | |

**Name:** SOLUTION

**Problem 1 (15 points)**

There have been seven recorded alien invasions of Earth:

(a) May 1902, six Vulcan ships landed in Fort Lauderdale.

(b) December 1928, twelve Klingon ships landed in Pensacola.

(c) March 1930, four Vulcan ships landed in Miami.

(d) July 1950, eleven Vulcan ships landed in Orlando.

(e) August 1992, two Klingon ships landed in St. Augustine.

(f) January 1993, eight Klingon ships landed in Daytona.

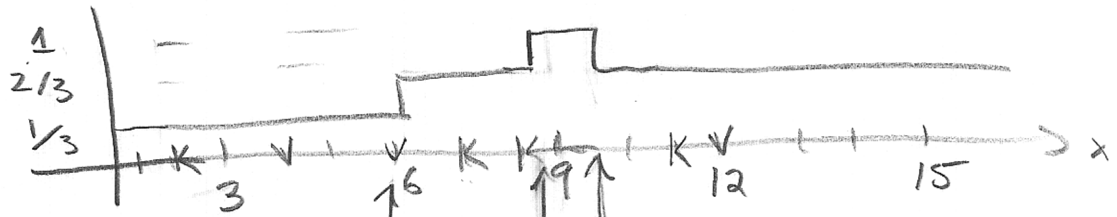
(g) May 2003, seven Klingon ships landed in Palm Beach.

The United Nations has commissioned you to create a Classifier of Invasions by Aliens (CIA). Your CIA should be a function defined by

$$f_{CIA}(x) \equiv \Pr\{\text{KLINGONS}|\text{Number of ships} = x\}$$

Draw $f_{CIA}(x)$ as a function of $x$, for $0 < x < 15$, using a **3-nearest-neighbor** rule to estimate the probability. You may assume that Klingons and Vulcans are the only alien races that exist, thus $\Pr\{\text{KLINGONS}|x\} = 1 - \Pr\{\text{VULCANS}|x\}$

**IMPORTANT: Specify the value of $x$ at each discontinuity.**




$\frac{12+7}{2} = \frac{19}{2}$

$\frac{4+8}{2} = 6$

$\frac{6+11}{2} = \frac{17}{2}$

**Problem 2 (30 points)**

A pelican fishes by sweeping its beak through the water. Each sweep catches many fish. The total weight of fish caught in a single sweep is an instance of a random variable, $X$, that is well described by a Gaussian mixture model:

$$p_X(x) = \sum_{k=1}^{2} c_k \mathcal{N}(x; \mu_k, \sigma_k^2)$$

Unfortunately, you don't know what are the correct values of the parameters $c_k$, $\mu_k$, and $\sigma_k$.

(a) You have received the following suggestions for the parameters. For each candidate set of parameters, say whether or not $p_X(x)$ would be a valid probability density if computed using this set of parameters; if not, say why not.

(i) Alice suggests $c_1 = 1, c_2 = 1, \mu_1 = 10, \mu_2 = 20, \sigma_1 = 10, \sigma_2 = 10$. Would $p_X(x)$ computed using this parameter set be a valid probability density? If not, why not?

No, because $c_1 + c_2 \neq 1$.

(ii) Barb suggests $c_1 = 0.1, c_2 = 0.9, \mu_1 = 0, \mu_2 = 20, \sigma_1 = 10, \sigma_2 = 10$. Would $p_X(x)$ computed using this parameter set be a valid probability density? If not, why not?

Yes.

(iii) Carol suggests $c_1 = 0.5, c_2 = 0.5, \mu_1 = 10, \mu_2 = 20, \sigma_1 = -10, \sigma_2 = 10$. Would $p_X(x)$ computed using this parameter set be a valid probability density? If not, why not?

No, because $\sigma_1 < 0$

(Actually: yes because $\sigma_1^2 > 0$ is also an acceptable answer!)

(b) You follow a pelican named Pete, and measure the weight of fish he retrieves on four consecutive dips, resulting in the following training dataset:

$$\{x_1, \ldots, x_4\} = \{5, 25, 15, 10\}$$

Using the parameter set $c_1 = 0.5, c_2 = 0.5, \mu_1 = 10, \mu_2 = 20, \sigma_1 = 10, \sigma_2 = 10$, compute $\gamma_k(x_t) = \Pr\{k^{\text{th}}\ \text{Gaussian}|x_t\}$ for $1 \le t \le 4$, $1 \le k \le 2$. You might find the table of Gaussian PDFs on page 2 of this exam to be useful.

$$\gamma_k(x_t) = \frac{c_k \mathcal{N}(x; \mu_k, \sigma_k^2)}{\sum_\ell c_\ell \mathcal{N}(x; \mu_\ell, \sigma_\ell^2)}$$

$$\mathcal{N}(5; 10, 10) = \frac{1}{\sqrt{2\pi \cdot 100}} e^{-\frac{1}{2}\left(\frac{5-10}{10}\right)^2} = \frac{1}{10}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0.5)^2}$$

$$= \frac{1}{10}(0.35) = 0.035$$

$$\mathcal{N}(5; 20, 10) = \frac{1}{10}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1.5)^2} = \frac{1}{10}(0.13) = 0.013$$

$$\mathcal{N}(25; 10, 10) = \frac{1}{10}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1.5)^2} = 0.013$$

$$\mathcal{N}(25; 20, 10) = 0.035$$

$$\mathcal{N}(15; 10, 10) = 0.035$$

$$\mathcal{N}(15; 10, 10) = 0.035$$

$$\mathcal{N}(10; 10, 10) = \frac{1}{10}(0.4) = 0.04$$

$$\mathcal{N}(10; 20, 10) = \frac{1}{10}(0.24) = 0.024$$

$$\gamma_1(x_1) = \frac{0.035}{0.035 + 0.013} \qquad \gamma_2(x_1) = \frac{0.013}{0.035 + 0.013}$$

$$\gamma_1(x_2) = \frac{0.013}{0.035 + 0.013} \qquad \gamma_2(x_2) = \frac{0.035}{0.013 + 0.035}$$

$$\gamma_1(x_3) = \frac{1}{2} \qquad \gamma_2(x_3) = \frac{1}{2}$$

$$\gamma_1(x_4) = \frac{0.04}{0.064} \qquad \gamma_2(x_4) = \frac{0.024}{0.064}$$

(c) Recall that the training data are

$$\{x_1, \ldots, x_4\} = \{5, 25, 15, 10\}$$

Suppose that, after a few iterations of EM, you wind up with the following gamma probabilities:

$$\{\gamma_2(x_1), \gamma_2(x_2), \gamma_2(x_3), \gamma_2(x_4)\} = \{0.1, 0.8, 0.6, 0.6\}$$

Find the re-estimated values of $c_2$, $\mu_2$, and $\sigma_2^2$ resulting from this iteration of EM.

$$\hat{c}_2 = \frac{\sum_{t=1}^{T} \gamma_2(x_t)}{T} = \frac{0.1 + 0.8 + 0.6 + 0.6}{4}$$

$$\hat{\mu}_2 = \frac{\sum_{t=1}^{T} \gamma_2(x_t)\, x_t}{\sum_{t=1}^{T} \gamma_2(x_t)} = \frac{0.1 \cdot 5 + 0.8 \cdot 25 + 0.6 \cdot 15 + 0.6 \cdot 10}{0.1 + 0.8 + 0.6 + 0.6}$$

$$\hat{\sigma}_2^2 = \frac{\sum_{t=1}^{T} \gamma_2(x_t)(x_t - \hat{\mu}_2)^2}{\sum_{t=1}^{T} \gamma_2(x_t)} = \frac{0.1(5-\hat{\mu}_2)^2 + 0.8(25-\hat{\mu}_2)^2 + 0.6(15-\hat{\mu}_2)^2 + 0.6(10-\hat{\mu}_2)^2}{0.1 + 0.8 + 0.6 + 0.6}$$

**Problem 3 (15 points)**

You're training an audiovisual bird classifier: based on measurements of the birdsong frequency ($f$) and the bird color ($c$), the bird is classified as a sparrow ($s = 1$) if and only if

$$\eta \ln p(c|s=1) + (1-\eta)\ln p(f|s=1) > \eta \ln p(c|s=0) + (1-\eta)\ln p(f|s=0)$$

In truth, all sparrows have pitch $f < 0.5$, and color $c < 0.5$, while all other birds have pitch $f > 0.5$ and color $c > 0.5$. Unfortunately, your training algorithm is broken, so it learned these distributions:

$$p(f|s=0) = \begin{cases} 1 & 0 \le f \le 1 \\ 0 & \text{else} \end{cases}, \quad p(f|s=1) = \begin{cases} 1 & 0 \le f \le 1 \\ 0 & \text{else} \end{cases}, \quad p(c|s=0) = \begin{cases} 1 & 0 \le c \le 1 \\ 0 & \text{else} \end{cases}$$

In fact, only one of the pdfs was learned to be non-uniform:

$$p(c|s=1) = \begin{cases} 2-2c & 0 \le c \le 1 \\ 0 & \text{else} \end{cases}$$

Despite these horrible training results, it is still possible to choose a value of $\eta$ so that your audiovisual fusion system has zero error. What value of $\eta$ gives your classifier zero error?

ZERO ERROR MEANS THAT:

$$\eta \ln p(c|s=1) + (1-\eta)\cancelto{0}{\ln p(f|s=1)} - \eta \cancelto{0}{\ln p(c|s=0)} - (1-\eta)\cancelto{0}{\ln p(f|s=0)} \begin{cases} > 0 & f < 0.5 \text{ and } c < 0.5 \\ < 0 & f > 0.5 \text{ and } c > 0.5 \\ \text{don't care} & \text{otherwise} \end{cases}$$

IN OTHER WORDS

$$\boxed{\eta \ln(2-2c) \quad \begin{matrix} > 0 & c < 0.5 \\ < 0 & c > 0.5 \end{matrix}}$$

BUT $\ln(2-2c) \quad \begin{matrix} > 0 & c < 0.5 \\ < 0 & c > 0.5 \end{matrix}$

SO ANY VALUE OF $\eta$ IS OK, $0 < \eta \le 1$

**Problem 4 (15 points)**

Good days and bad days follow each other with the following probabilities:

| $q_{t-1}$ | $p(q_t = G \vert q_{t-1} = \cdot)$ | $p(q_t = B \vert q_{t-1} = \cdot)$ |
|---|---|---|
| G | 0.7 | 0.3 |
| B | 0.4 | 0.6 |

In winter in Champaign, the temperature on a good day is Gassian with mean $\mu_G = 50$, $\sigma_G = 20$. The temperature on a bad day is Gaussian with mean $\mu_B = 10$, $\sigma_G = 20$. A particular sequence of days has temperatures

$$\{x_1 = 10, x_2 = 20, x_3 = 30\}$$

What is the probability $p(X|q_1 = B)$, the probability of seeing this sequence of temperatures given that the first day was a bad day?

$$p(x_1, x_2, x_3 \mid q_1 = B) = \sum_{j=1}^{2} \alpha_3(j)$$

$$\alpha_t(i) = p(x_1, \ldots, x_t, q_t = i \mid q_1 = B)$$

$$\alpha_1(B) = b_B(10) = N(10; 10, 20) = \tfrac{1}{20} N(0) = \tfrac{0.4}{20} = \tfrac{1}{50}$$

$$\alpha_1(G) = 0$$

$$\alpha_2(B) = \alpha_1(B)\, a_{BB}\, b_B(20) = \left(\tfrac{1}{50}\right)(0.6) N(20; 10, 20)$$
$$= \left(\tfrac{1}{50}\right)(0.6)\left(\tfrac{1}{20} \cdot N(0.5)\right) = (0.4)(0.6)\left(\tfrac{1}{20}\right)(0.35)$$

$$\alpha_2(G) = \alpha_1(B)\, a_{BG}\, b_G(20) = \left(\tfrac{1}{50}\right)(0.4) N(20; 50, 20)$$
$$= \left(\tfrac{1}{50}\right)(0.4)\left(\tfrac{1}{20}\right)(N(-1.5)) = \left(\tfrac{1}{50}\right)(0.4)\left(\tfrac{1}{20}\right)(0.13)$$

$$\alpha_3(B) = \sum_{j=1}^{2} \alpha_2(j)\, a_{jB}\, b_B(30)$$
$$= \left(\left(\tfrac{1}{50}\right)(0.6)\left(\tfrac{1}{20}\right)(0.35)(0.6)\left(\tfrac{1}{20}\right) N(1)\right.$$
$$\left. + \left(\tfrac{1}{50}\right)(0.4)\left(\tfrac{1}{20}\right)(0.13)(0.3)\left(\tfrac{1}{20}\right) N(1)\right)$$

$$\alpha_3(G) = \left(\left(\tfrac{1}{50}\right)(0.6)\left(\tfrac{1}{20}\right)(0.35)(0.4)\left(\tfrac{1}{20}\right) N(1)\right.$$
$$\left. + \left(\tfrac{1}{50}\right)(0.4)\left(\tfrac{1}{20}\right)(0.13)(0.7)\left(\tfrac{1}{20}\right) N(1)\right)$$

$$p(X \mid q_1 = B) = \alpha_3(B) + \alpha_3(G)$$
$$= \left(\tfrac{1}{50}\right)\left(\tfrac{1}{20}\right)\left(\tfrac{1}{20}\right)(0.24)\Big((0.6)(0.35)(0.6) + (0.4)(0.13)(0.3)$$
$$+ (0.6)(0.35)(0.4) + (0.4)(0.13)(0.7)\Big)$$

**Problem 5 (25 points)**

Suppose that

$$\begin{aligned} a_{ij} &= p(q_t = j | q_{t-1} = i) \\ b_j(x_t) &= p(x_t | q_t = j) \\ g_t &= p(x_t | x_1, \ldots, x_{t-1}) \end{aligned}$$

And define the scaled forward algorithm to compute

$$\tilde{\alpha}_t(i) = p(q_t = i | x_1, \ldots, x_t) = \frac{p(x_t, q_t = i | x_1, \ldots, x_{t-1})}{g_t} = \frac{p(x_1, \ldots, x_t, q_t = i)}{g_1 g_2 \cdots g_t}$$

(a) Devise an algorithm to iteratively compute $g_t$ and $\tilde{\alpha}_t(i)$. Fill in the right-hand side of each equation, using only the terms $a_{jk}$, $b_j(x_\tau)$, $g_\tau$, and $\tilde{\alpha}_\tau(j)$ for $1 \le j \le N$, $1 \le k \le N$, $1 \le \tau \le t$.

(i) **INITIALIZE:** $g_1 = p(x_1) = \sum_{j=1}^{N} \pi_j b_j(x_1)$

(ii) **INITIALIZE:** $\tilde{\alpha}_1(i) = \dfrac{\pi_i b_i(x_1)}{g_1}$

(iii) **ITERATE:** $g_t = \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{\alpha}_{t-1}(i) a_{ij} b_j(x_t)$

(iv) **ITERATE:** $\tilde{\alpha}_t(i) = \dfrac{\sum_{j=1}^{N} \tilde{\alpha}_{t-1}(j) a_{ji} b_i(x_t)}{g_t}$

(v) **TERMINATE:** $p(X) = \prod_{t=1}^{T} g_t$

(b) Suppose $\beta_t(i) = p(x_{t+1}, \ldots, x_T | q_t = i)$. Then

$$\tilde{\alpha}_t(i)\beta_t(i) = p(f|g)$$

for some list of variables $f$, and some other list of variables $g$. Specify what variables should be included in each of these two lists.

$f = \{\ q_t = i,\ x_{t+1}, \ldots, x_T\ \}$

$g = \{\ x_1, \ldots, x_t\ \}$

$$\tilde{\alpha}_t(i)\beta_t(i) = p(q_t = i | x_1, \ldots, x_t)\, p(x_{t+1}, \ldots, x_T | q_t = i)$$

$$= p(q_t = i, x_{t+1}, \ldots, x_T | x_1, \ldots, x_t)$$