UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Fall 2017

CONFLICT EXAM 1

Tuesday, October 3, 2017

- This is a CLOSED BOOK exam. You may use one sheet (front and back) of hand-written notes.
- No calculators are permitted. You need not simplify explicit numerical expressions.
- There are a total of 40 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score
1	
2	
3	
4	
Total	

Name: ____

Possibly Useful Formulas

Minkowski Norm

$$\|\vec{x}-\vec{\mu}\|_p = \left(|x_1-\mu_1|^p + \ldots + |x_D-\mu_D|^p\right)^{1/p}$$

Gaussians

$$\mathcal{N}\left(\vec{x};\vec{\mu},\Sigma\right) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

$$\begin{aligned}
\Sigma &= U\Lambda U^T \\
\Sigma^{-1} &= U\Lambda^{-1}U^T \\
U^T\Sigma U &= \Lambda \\
U^T U &= I
\end{aligned}$$

Mahalanobis Distance and PCA

$$d^2_\Sigma(\vec{x},\vec{\mu}) = (\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu}) = \vec{y}^T\Lambda^{-1}\vec{y}$$

$$\vec{y} = U^T(\vec{x}-\vec{\mu})$$

Bayesian Classifier

$$\hat{y} = \arg\max p_{Y|\vec{X}}(y|\vec{x})$$

Problem 1 (10 points)

Suppose you have a dataset including the vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

(a) Consider the following function $D(\vec{x}, \vec{y})$. Is it a distance? Why or why not?

$$D(\vec{x}, \vec{y}) = |x_1 + x_2 + x_3 - y_1 - y_2 - y_3|$$

(b) Find a diagonal matrix Σ such that $d_\Sigma^2(\vec{x}, \vec{y}) > d_\Sigma^2(\vec{x}, \vec{z})$. Express your answer in terms of the variables $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$.

Problem 2 (10 points)

Define $\Phi(z)$ as follows:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

Suppose $\vec{X} = [X_1, X_2]^T$ is a Gaussian random vector with mean and covariance given by

$$\vec{\mu} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

(a) Define $f_{\vec{X}}(\vec{x})$, to be the pdf of $\vec{X}$ evaluated at $\vec{x} = [x_1, x_2]^T$. Sketch, on the (x_1, x_2) plane, the set of points such that

$$f_{\vec{X}}(\vec{x}) = \frac{1}{4\pi} e^{-\frac{1}{8}(x_2-2)^2}$$

(b) In terms of $\Phi(z)$, find the probability $\Pr\{2 < X_1\}$.

Problem 3 (10 points)

Suppose that a particular covariance matrix Σ has the following eigenvector matrix, U, and eigenvalue matrix, Λ:

$$U = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $\vec{y}(\vec{x}) = \begin{bmatrix} y_1(\vec{x}) \\ y_2(\vec{x}) \end{bmatrix} = U^T\vec{x}$ be the principal components of a vector space $\vec{x}$.

(a) Plot the set of vectors $\vec{x}$ such that $\vec{x}^T\Sigma^{-1}\vec{x} = 1$

(b) Find the principal component representation of the following vector:

$$\vec{x} - \vec{\mu} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Problem 4 (10 points)

Suppose that, for a particular classification problem, you have the following nine data points $\vec{x}_n$ and their labels y_n:

$$X = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}, \quad Y = [0,1,0,1,1,1,0,1,0] \tag{1}$$

(a) Plot the boundaries of the nearest-neighbor classifier, for the region $-2 \le x_1 \le 2, -2 \le x_2 \le 2$.

(b) Suppose now that $f_{\vec{X}|Y}(\vec{x}|0)$ and $f_{\vec{X}|Y}(\vec{x}|1)$ are both zero-mean Gaussian pdfs, with the covariance matrices Σ_0 and Σ_1 respectfully, where

$$\Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Define η to be the odds ratio, $\eta = p_Y(0)/p_Y(1)$. Find a value of η such that a Bayesian classifier correctly labels all of the training tokens given in Eq. (1).