UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Principles of Signal Analysis
Spring 2015

EXAM 3

Monday, May 11, 2015

- This is a CLOSED BOOK exam. You may use one sheet (front and back) of hand-written notes.
- No calculators are permitted. You need not simplify explicit numerical expressions.
- There are a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score
1	
2	
3	
4	
5	
Total	

Name: ____

Possibly Useful Formulas

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \leftrightarrow X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \leftrightarrow X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$h[n] = \frac{\sin\omega_c n}{\pi n} \leftrightarrow H(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$u[n] - u[n-N] \leftrightarrow e^{-j\frac{\omega(N-1)}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$\delta[n] \leftrightarrow 1$$

$$e^{j\alpha n} \leftrightarrow 2\pi\delta(\omega-\alpha)$$

$$\sum_{\ell=-\infty}^{\infty}\delta[n-\ell T_0] \leftrightarrow \left(\frac{2\pi}{T_0}\right)\sum_{k=1}^{T_0-1}\delta\left(\omega - \frac{2\pi k}{T_0}\right)$$

$$S = \sum_{k=1}^{n}(\vec{x}_k - \vec{m})(\vec{x}_k - \vec{m})^T$$

$$\mathcal{N}(\vec{x};\vec{\mu},\Sigma) = \frac{1}{(2\pi)^{\dim(\vec{x})/2}\det(\Sigma)^{1/2}}e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

θ	$\cos\theta$	$\sin\theta$	$e^{j\theta}$
0	1	0	1
$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/2 + j/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2 + j\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$	$1/2 + j\sqrt{3}/2$
$\pi/2$	0	1	j
π	-1	0	-1
$3\pi/2$	1	-1	$-j$
2π	1	0	1

Problem 1 (20 points)

The Minkowski $\frac{1}{2}$-norm is a measure of dissimilarity between vectors $\vec{x} = [x_1, \ldots, x_d]^T$ and $\vec{y} = [y_1, \ldots, y_d]^T$. It is defined as

$$d_{\frac{1}{2}}(\vec{x}, \vec{y}) = \left(\sum_{\ell=1}^{d} \sqrt{|x_\ell - y_\ell|} \right)^2$$

Does the Minkowski $\frac{1}{2}$-norm satisfy the triangle inequality? If so, prove it. If not, provide a counter-example.

Problem 2 (20 points)

As you know, in any given vector space, it's possible to define an infinite number of different Mahalanobis distances. Consider the Mahalanobis distance measures $d_a(\vec{x},\vec{y})$ and $d_b(\vec{x},\vec{y})$, parameterized, respectively, by the covariance matrices

$$\Sigma_a = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \vdots \\ \vdots & 0 & \dots & 0 \\ 0 & \dots & 0 & a_d \end{bmatrix}, \quad \Sigma_b = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \vdots \\ \vdots & 0 & \dots & 0 \\ 0 & \dots & 0 & b_d \end{bmatrix}$$

Your friend Amit wishes to define a third dissimilarity measure, as

$$d_c(\vec{x},\vec{y}) = \sqrt{\frac{1}{2}\left(d_a^2(\vec{x},\vec{y}) + d_b^2(\vec{x},\vec{y})\right)}$$

Is $d_c(\vec{x},\vec{y})$ a Mahalanobis distance? If so, find the elements of the covariance matrix Σ_c in terms of the variables $a_1,\dots,a_d,b_1,\dots,b_d$. If not, demonstrate that no such covariance matrix exists.

Problem 3 (20 points)

For this problem, define the cepstral distance between signals $x[n]$ and $y[n]$ to be

$$d(x[n], y[n]) = \sqrt{\sum_{m=0}^{N-1} (\hat{x}[m] - \hat{y}[m])^2}$$

where the cepstrum is defined as

$$\hat{x}[m] = \frac{1}{N} \sum_{k=0}^{N-1} \max\left(0, \log|X[k]|\right) e^{j2\pi km/N}$$

and the DFT is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

In terms of the amplitudes A and B, the frequencies k and ℓ, and the phase terms θ and ϕ, find the cepstral distance between the following two signals:

$$x[n] = A\cos\left(\frac{2\pi kn}{N} + \theta\right), \quad y[n] = B\cos\left(\frac{2\pi \ell n}{N} + \phi\right)$$

Problem 4 (20 points)

Consider the signal

$$x[n] = \cos\left(\frac{2\pi n}{12}\right) + A\cos\left(\frac{2\pi n}{8}\right)$$

Assume a frame length of 48 samples. As a function of the parameter A, what is the zero-crossing rate of this signal?

Problem 5 (20 points)

A color histogram is a matrix whose element $N[k_R, k_B]$, $1 \leq k_R \leq 10$ and $1 \leq k_B \leq 10$, is the number of pixels for which $0.1(k_R - 1) \leq s_R[m,n] < 0.1k_R$ and $0.1(k_B - 1) \leq s_B[m,n] < 0.1k_B$. The redshift and blueshift of pixel $\vec{y}[m,n] = [r[m,n], g[m,n], b[m,n]]^T$ are defined as

$$s_R[m,n] = \frac{r[m,n]}{r[m,n] + g[m,n] + b[m,n]}, \quad s_B[m,n] = \frac{b[m,n]}{r[m,n] + g[m,n] + b[m,n]}$$

Define the color-histogram-distance between two images to be

$$d(\vec{y_1}, \vec{y_2}) = \sqrt{\sum_{k_R=1}^{10} \sum_{k_B=1}^{10} |N_1[k_R, k_B] - N_2[k_R, k_B]|^2}$$

where $N_1[k_R, k_B]$ and $N_2[k_R, k_B]$ are the color histograms resulting from images $\vec{y_1}[m,n]$ and $\vec{y_2}[m,n]$), respectively. Consider four images, two of constant color, and two with broad stripes:

$$\vec{y_1}[m,n] = \begin{bmatrix} 253 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{y_2}[m,n] = \begin{bmatrix} 1 \\ 1 \\ 253 \end{bmatrix}$$

$$\vec{y_3}[m,n] = \begin{cases} \vec{y_1}[m,n] & 0 \leq m \leq 4 \\ \vec{y_2}[m,n] & 5 \leq m \leq 9 \end{cases}, \qquad \vec{y_4}[m,n] = \begin{cases} \vec{y_2}[m,n] & 0 \leq m \leq 4 \\ \vec{y_1}[m,n] & 5 \leq m \leq 9 \end{cases}$$

Assume that all four are 10×10-pixel images (only defined for $0 \leq m \leq 9$, $0 \leq n \leq 9$). Find the numerical values of $d(\vec{y_1}, \vec{y_2})$ and $d(\vec{y_3}, \vec{y_4})$. For partial credit, in case your numerical computation is wrong, state which one you believe to be larger, and why.