ECE 417 Lecture 3: 1-D Gaussians

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Cumulative Distribution Function (CDF)

A “cumulative distribution function” (CDF) specifies the probability that random variable X takes a value less than θ:

$$F_X(x) = Pr\{X \leq x\}$$

Probability Density Function (pdf)

A "probability density function" (pdf) is the derivative of the CDF:

$$f_X(x) = \frac{d}{dx} F_X(x)$$

That means, for example, that the probability of getting an X in any interval $a < X \leq b$ is:

$$\Pr\{a < X \leq b\} = \int_a^b f_X(x) dx$$

Example: Uniform pdf

The rand() function in most programming languages simulates a number uniformly distributed between 0 and 1, that is,

$$f_X(x) = \begin{cases} 1 : & 0 \leq x < 1 \\ 0 : & otherwise \end{cases}$$

Suppose you generated 100 random numbers using the rand() function.

- How many of the numbers would be between 0.5 and 0.6?
- How many would you expect to be between 0.5 and 0.6?
- How many would you expect to be between 0.95 and 1.05?

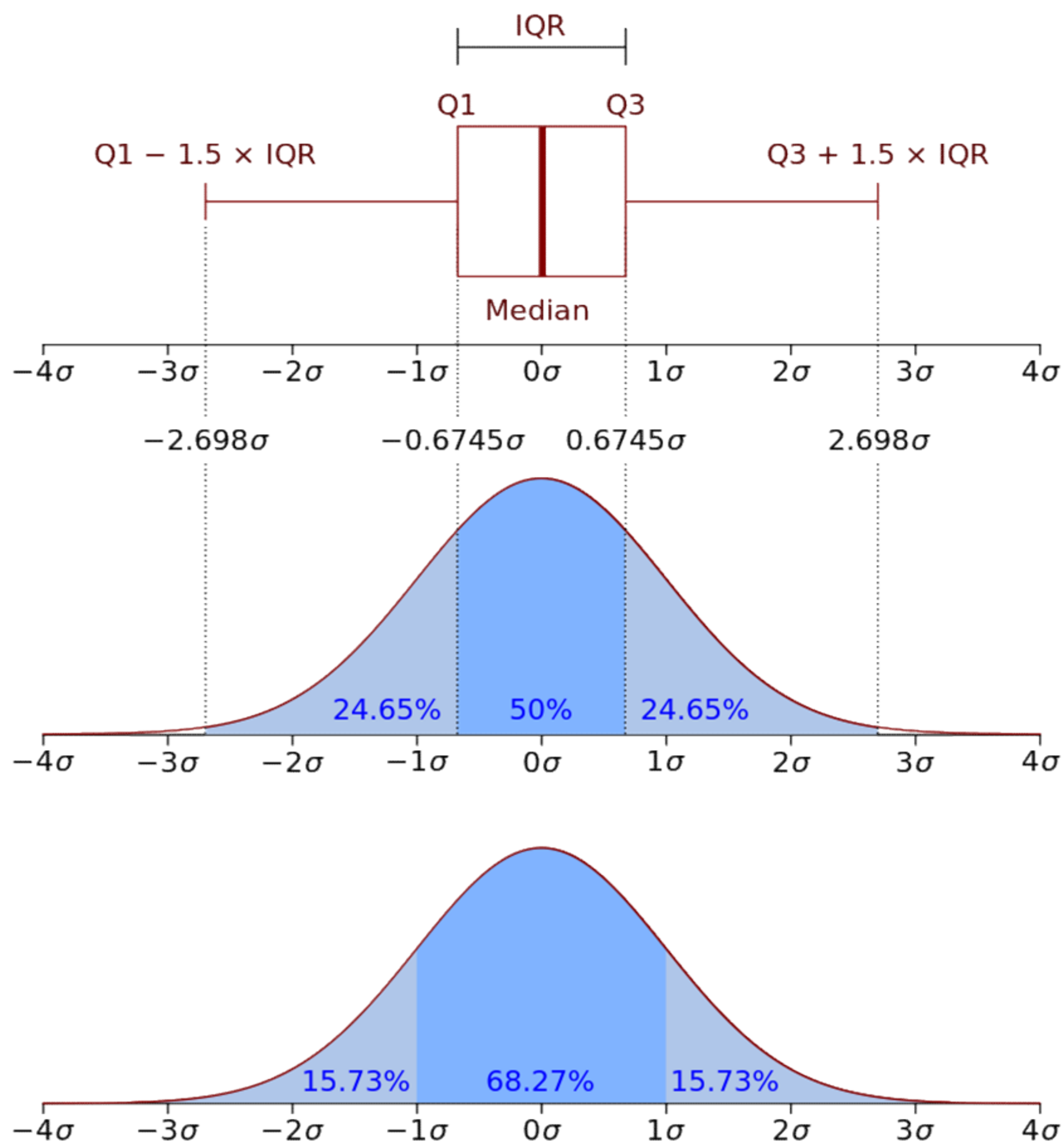
Gaussian (Normal) pdf

Gauss considered this problem: under what circumstances does it make sense to estimate the mean of a distribution, μ, by taking the average of the experimental values, $\mathrm{m} = \frac{1}{n}\sum_{i=1}^{n} x_i$?

He demonstrated that m is the maximum likelihood estimate of μ if

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Gaussian pdf



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Unit Normal pdf

Suppose that X is normal with mean μ and standard deviation σ (variance σ^2):

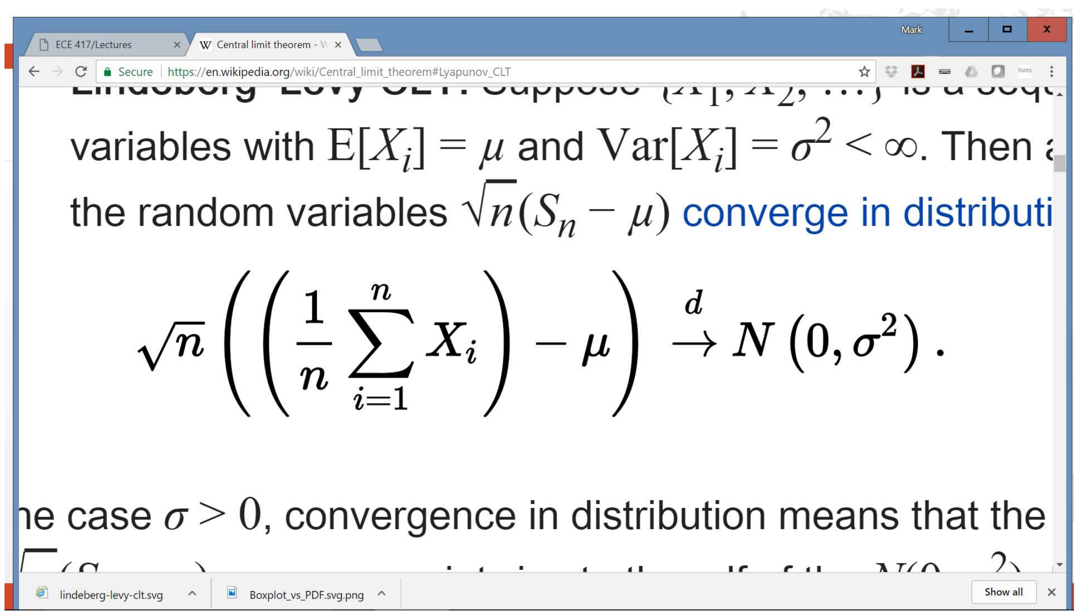
$$f_X(x) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Then $U = \left(\frac{X-\mu}{\sigma}\right)$ is normal with mean 0 and standard deviation 1:

$$f_U(u) = \mathcal{N}(u; 0,1\ ) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$$

Central Limit Theorem

The Gaussian pdf is important because of the Central Limit Theorem. Suppose X_i are i.i.d. (independent and identically distributed), each having mean μ and variance σ^2. Then



Example: the sum of uniform random variables

Suppose that X_i are i.i.d. unit uniform random variables, i.e.,

$$f_{X_i}(x_i) = \begin{cases} 1 \ : \ \ 0 \le x_i < 1 \\ 0 : otherwise \end{cases}$$

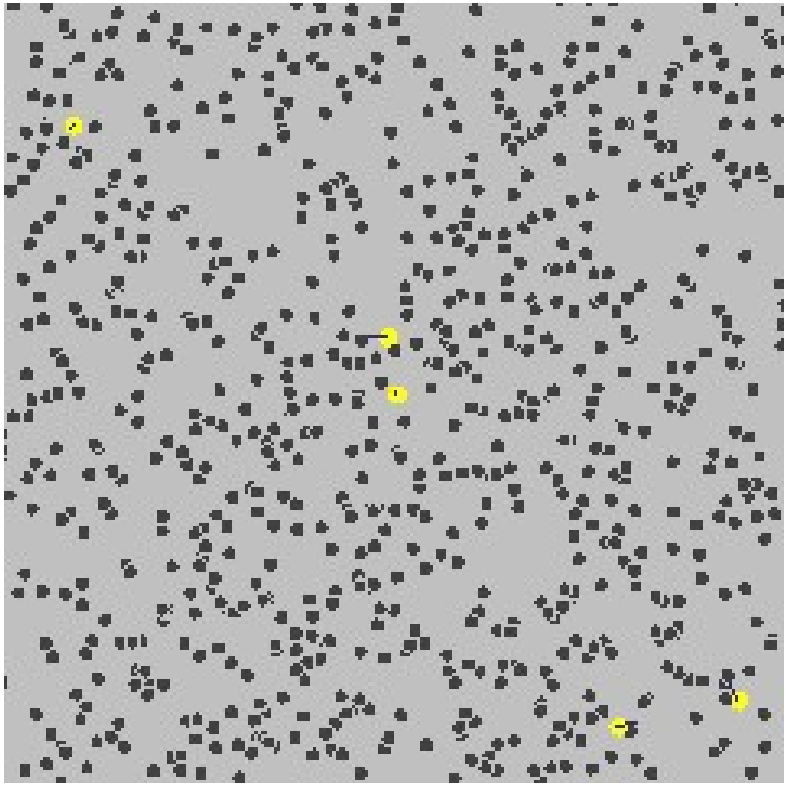
Consider the sum, $S = \sum_{i=1}^{n} X_i$. The CDF is

$$F_S(s) = Pr\{X_1 + \cdots + X_n \le s\}$$
$$= \int_{x_1+\cdots+x_n \le s} 1 \, dx_1 \dots dx_n$$

Brownian motion

The Central Limit Theorem matters because Einstein showed that the movement of molecules, in a liquid or gas, is the sum of n i.i.d. molecular collisions.

In other words, the position after t seconds is Gaussian, with mean 0, and with a variance of Dt, where D is some constant.



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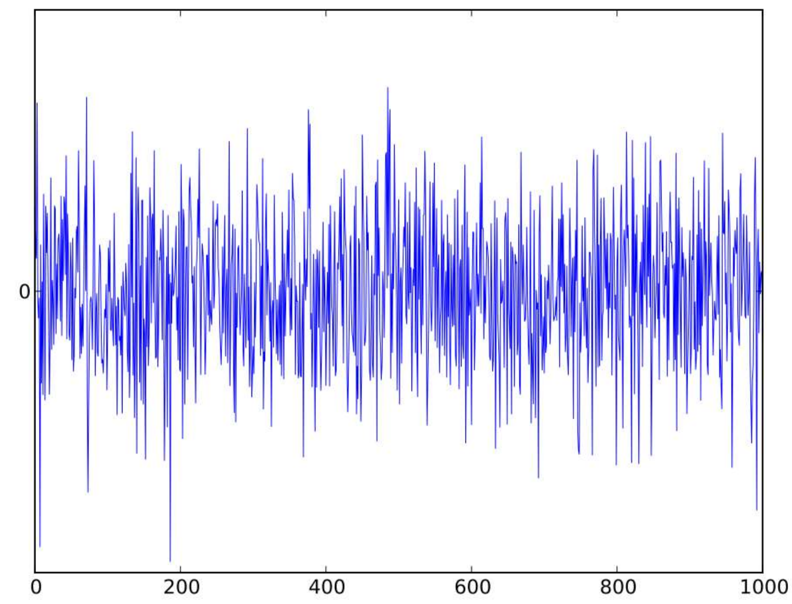
Gaussian Noise

- Sound = air pressure fluctuations caused by velocity of air molecules
- Velocity of warm air molecules without any external sound source = Gaussian

Therefore:

- Sound produced by warm air molecules without any external sound source = Gaussian noise
- Electrical signals: same.



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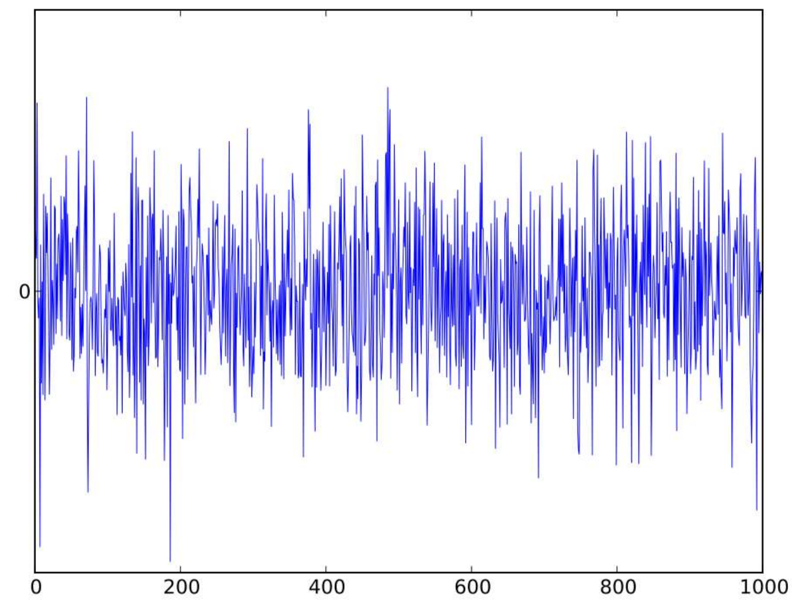
White Noise

- White Noise = noise in which each sample of the signal, x[n], is i.i.d.
- Why “white”? Because the Fourier transform, $X(\omega)$, is a zero-mean random variable whose variance is independent of frequency (“white”)
- ***Gaussian White Noise***: x[n] are i.i.d. and Gaussian



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Vector of Independent Gaussian Variables

Suppose we have a frame containing N samples from a Gaussian white noise process, $x_1, \ldots, x_N$. Let's stack them up to make a vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

This whole frame is random. In fact, we could say that $\vec{x}$ is a sample value for a Gaussian random vector called $\vec{X}$, whose elements are $X_1, \ldots, X_N$:

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}$$

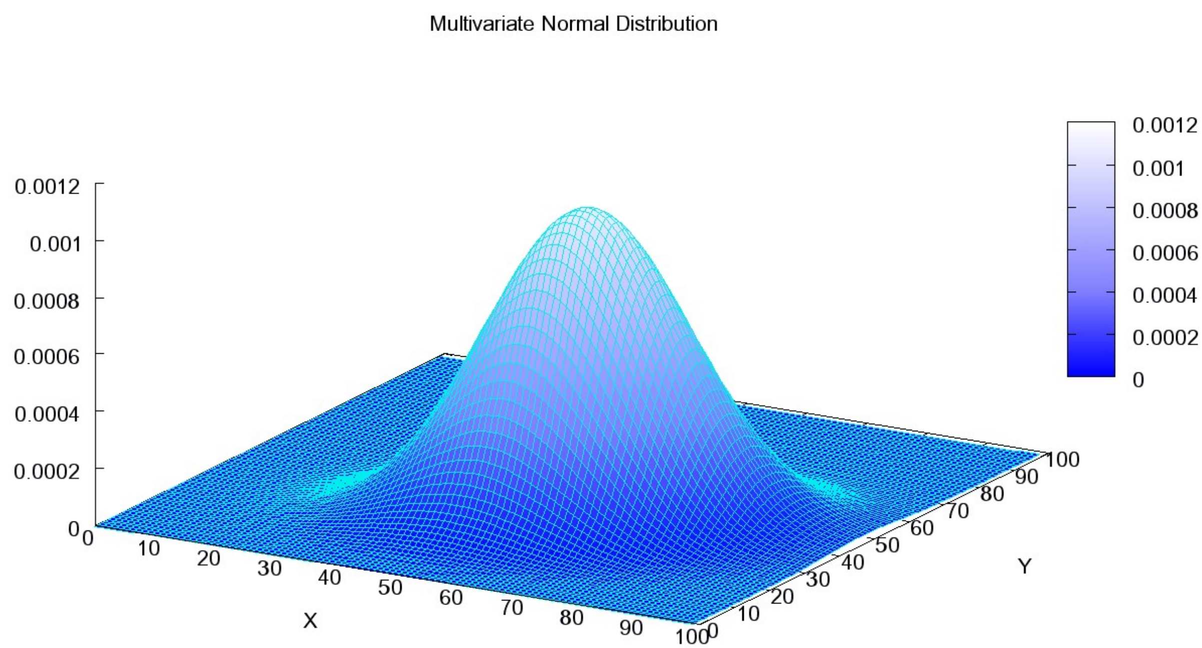
Vector of Independent Gaussian Variables

Suppose that the N samples are i.i.d., each one has the same mean, μ, and the same variance, σ^2. Then the pdf of this random vector is

$$f_{\vec{X}}(\vec{x}) = \mathcal{N}(\vec{x}; \vec{\mu}, \sigma^2 I) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x_n-\mu}{\sigma}\right)^2}$$

Vector of Independent Gaussian Variables

For example, here's an example from Wikipedia with mean of 50 and standard deviation of about 12.



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Summary

- CDF = probability that X is less than or equal to x
- pdf = derivative of the CDF
- Gaussian: pdf is proportional to $\exp(-x^2)$
- CLT: if you average N random variables of any kind, the pdf of the average converges to a Gaussian
- Brownian motion, e.g., of air molecules in warm air, is the average of many random impacts = Gaussian movement
- White noise = i.i.d. random samples. Gaussian white noise: samples are Gaussian and i.i.d.
- Gaussian vector with independent elements: pdf of the vector = product of the pdfs of the elements