

Lecture 23: Autocorrelation

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ECE 401: Signal and Image Analysis

- ① Autocorrelation and covariance
- ② Autocorrelation and convolution
- ③ Autocorrelation and power spectrum
- ④ Examples: white noise and pink noise
- ⑤ Summary

Outline

- 1 Autocorrelation and covariance
- 2 Autocorrelation and convolution
- 3 Autocorrelation and power spectrum
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Autocorrelation

In this class we will define autocorrelation to be

$$r_x[n] = \sum_{m=-\infty}^{\infty} x[m]x^*[m-n]$$

... where, if $x[n]$ is real, then $x^*[n] = x[n]$.

Autocorrelation of a finite length signal

Suppose that $x[n] = 0$ unless $0 \leq n < N$. Then

$$r_x[n] = \sum_{m=n}^{N-1} x[m]x^*[m-n]$$

In that case, we can say that the “expected value” of the product is approximately

$$\frac{1}{N-n} \sum_{m=n}^{N-1} x[m]x^*[m-n] \approx E[x[m]x^*[m-n]]$$

Expectation

It's often useful to talk about a noisy signal as being the sum of:

- A deterministic (non-random) expected value, $\mu_x[n]$, plus
- ... a zero-mean random signal $v[n]$:

$$x[n] = \mu_x[n] + v[n]$$

$$E[x[n]] = \mu_x[n]$$

$$E[v[n]] = 0$$

- **Case 1: Denoising:** Assume that $\mu_x[n]$ exists, but we don't know it
- **Case 2: Normalization:** Assume that $\mu_x[n] \approx \frac{1}{N} \sum_{n=0}^{N-1} x[n]$, and just subtract it

Covariance and Variance

- The **covariance** between samples $x[m]$ and $x[m - n]$ is

$$\rho_x[m, m - n] = E [(x[m] - \mu_x[m])(x[m - n] - \mu_x[m - n])^*]$$

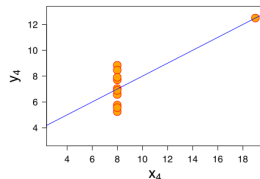
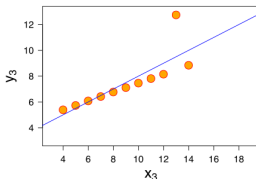
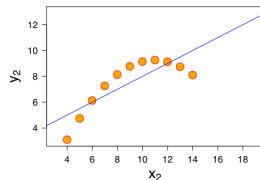
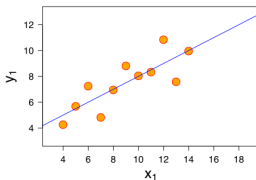
- The **variance** of sample $x[m]$ is

$$\sigma_x^2[m] = \rho_x[m, m] = E [|x[m] - \mu_x[m]|^2]$$

Why is covariance interesting?

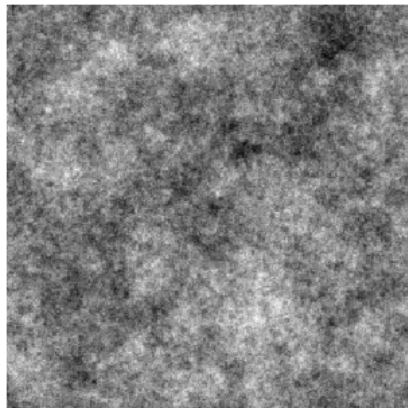
- Covariance is interesting because it tells us how predictable each sample is.
- For example, suppose you know $x[m - n]$ and want to predict $x[m]$. The minimum-mean-squared-error linear prediction is

$$\hat{x}[m] = \mu_x[m] + \frac{\rho_x[m, m - n]}{\sigma_x^2[m - n]} (x[m - n] - \mu_x[m - n])$$



- Examples of the formula $y \approx \mu_y + \frac{\rho(x,y)}{\sigma_x^2} (x - \mu_x)$
- In the same way, we could predict $x[n+1]$ from $x[n]$

- Each pixel in this image has positive covariance with its neighbors
- Each pixel is random, but generally similar to its neighbor
- You can predict a pixel by guessing that it will resemble its neighbor



Autocorrelation and covariance

- So what's the relationship?
- Autocorrelation:

$$\frac{r_x[n]}{N-n} = \frac{1}{N-n} \sum_{m=n}^{N-1} x[m]x^*[m-n] \approx E[x[m]x^*[m-n]]$$

- Covariance:

$$\rho_x[m, m-n] = E[(x[m] - \mu_x[m])(x[m-n] - \mu_x[m-n])^*]$$

Autocorrelation and covariance

Autocorrelation and covariance are the same thing

$$\left(\frac{1}{N-n} \sum_{m=n}^{N-1} x[m]x^*[m-n] \approx E[x[m]x^*[m-n]] \approx E[(x[m] - \mu_x[m])(x[m-n] - \mu_x[m-n])^*] \text{ if } x[n] \text{ is:} \right.$$

- 1 **Wide-sense stationary:** $\rho_x[m, m-n]$ depends only on the delay (n), not the time at which you measure it (m)
- 2 **Zero-mean:** $\mu_x[m] = 0$

Under those circumstances,

$$\begin{aligned} \rho_x[m, m-n] &= E[x[m]x^*[m-n]] \\ &\approx \frac{1}{N-n} r_x[n] \end{aligned}$$

What kinds of signals are stationary?

Real signals are never stationary, but for denoising purposes, we can often break a signal down into

$$x[n] = \mu_x[n] + v[n]$$

- Assume $\mu_x[n]$ is some expected signal that varies with time
- Assume $v[n]$ is a stationary zero-mean noise

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Autocorrelation is flipped convolution

Remember the formula for convolution:

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

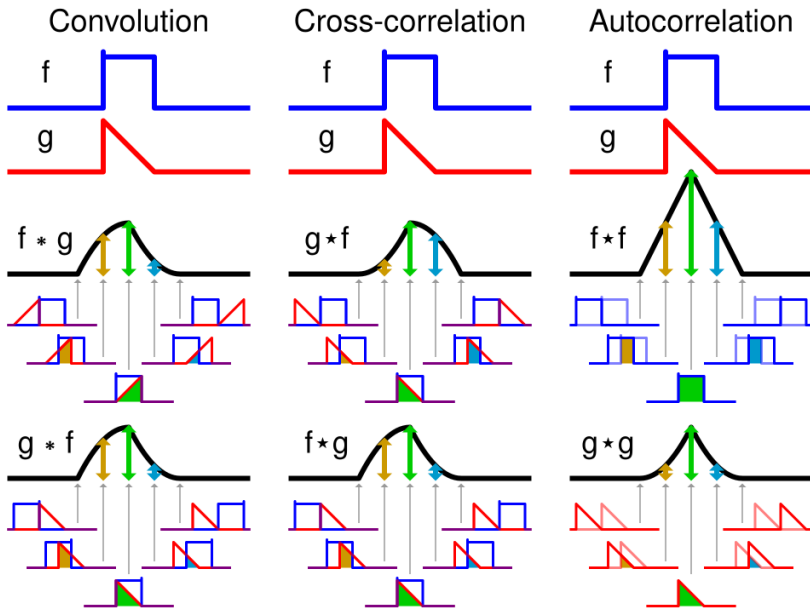
Compare to the formula for autocorrelation:

$$r_x[n] = \sum_{m=-\infty}^{\infty} x[m]x^*[m-n]$$

Lo and behold:

$$r_x[n] = x[n] * x^*[-n]$$

Autocorrelation is flipped convolution



Try the quiz!

Go to the course web page and try the quiz!

Energy is $r_x[0]$

- Energy is $r_x[0]$:

$$\mathcal{E}_x = \sum_{m=-\infty}^{\infty} |x[m]|^2 = r_x[0]$$

- Power = Variance = $\rho_x[0, 0] = \frac{1}{N} E[r_x[0]]$:

$$\sigma_x^2 = E[|x - \mu_x|^2] = \frac{1}{N} E[r_x[0]]$$

Autocorrelation is less than or equal to energy

$$\sum_{m=n}^{N-1} x[m]x^*[m-n] \leq \sum_{m=0}^{N-1} |x[m]|^2$$

And therefore

$$r_x[m] \leq r_x[0]$$

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Fourier transform of time-flipped $x[n]$

What is the Fourier transform of $x^*[-n]$?

$$\begin{aligned}\mathcal{F}\{x^*[-n]\} &= \sum_{m=-\infty}^{\infty} x^*[-m]e^{-j\omega m} \\&= \left(\sum_{m=-\infty}^{\infty} x[-m]e^{j\omega m} \right)^* \\&= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right)^*, \quad n = -m \\&= X^*(\omega)\end{aligned}$$

So the DTFT of $x^*[-n]$ is just $X^*(\omega)$!

Fourier transform of the autocorrelation

What is the Fourier transform of $r_x[n]$?

$$\begin{aligned}\mathcal{F}\{r_x[n]\} &= \mathcal{F}\{x[n] * x^*[-n]\} \\ &= \mathcal{F}\{x[n]\} \times \mathcal{F}\{x^*[-n]\} \\ &= X(\omega)X^*(\omega) \\ &= |X(\omega)|^2\end{aligned}$$

Power spectrum

- The **power spectrum** is defined to be $E [|X(\omega)|^2]$. Since the Fourier transform and expectation are both linear, we can swap their order, so that

$$\begin{aligned} E [|X(\omega)|^2] &= E [\mathcal{F} \{r_x[n]\}] \\ &= \mathcal{F} \{E [r_x[n]]\} \end{aligned}$$

- The **short-time power spectrum** (the one we'll deal with most often in this class) is just the DTFT of the autocorrelation of a short-time signal:

$$\mathcal{F} \{r_x[n]\} = |X(\omega)|^2$$

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Example: White Noise

- For example, **white noise** is a signal whose samples are completely uncorrelated with each other:

$$\begin{aligned} E[r_x[n]] &= E[x[m]x^*[m-n]] \\ &= \delta[n] \end{aligned}$$

- This is called white noise because its Fourier transform has equal energy at all frequencies, like white light:

$$\begin{aligned} E[|X(\omega)|^2] &= \mathcal{F}\{\delta[n]\} \\ &= 1 \end{aligned}$$

Example: Pink Noise

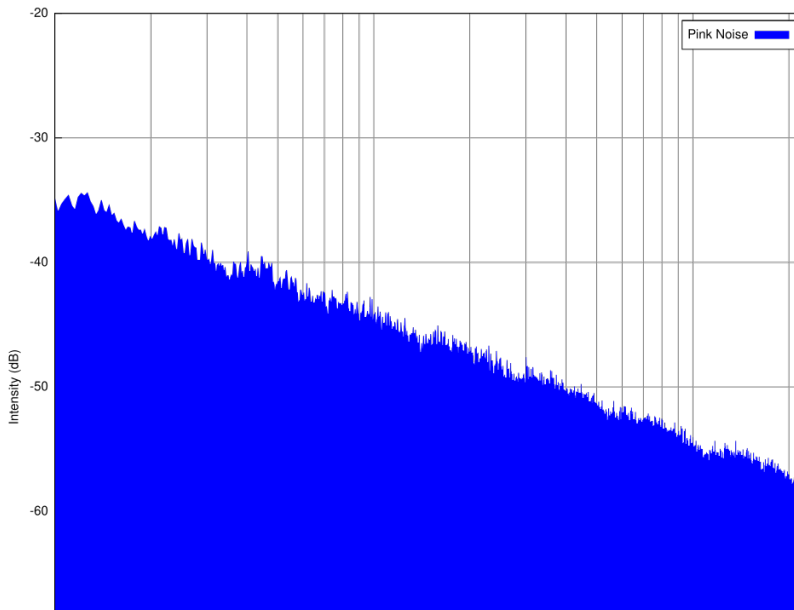
- **Pink Noise** is a signal whose samples have a small positive correlation with each other. For example:

$$E[r_x[n]] = \begin{cases} 1 & n = 0 \\ r/2 & |n| = 1 \\ 0 & |n| > 1 \end{cases}$$

- It's called **pink noise** because it has more energy at low frequencies, like red light:

$$\begin{aligned} E[|X(\omega)|^2] &= \mathcal{F}\{E[r_x[n]]\} \\ &= 1 + r \cos(\omega) \\ &= \begin{cases} 1 + r & \omega = 0 \\ 1 & \omega = \frac{\pi}{2} \\ 1 - r & \omega = \pi \end{cases} \end{aligned}$$

Spectrum of pink noise



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Summary

- Autocorrelation:

$$r_x[n] = \sum_{m=-\infty}^{\infty} x[m]x^*[m-n]$$

- If the signal is stationary and zero-mean, then autocorrelation is a scaled estimate of covariance:

$$\rho_x[m, m-n] = E[(x[m] - \mu_x[m])(x[m-n] - \mu_x[m-n])^*]$$

- Autocorrelation is flipped convolution:

$$r_x[n] = x[n] * x^*[-n]$$

- Power spectrum is the DTFT of autocorrelation:

$$\mathcal{F}\{r_x[n]\} = |X(\omega)|^2$$