

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 SIGNAL PROCESSING
EXAM 1

October 5, 2026

- This is a **CLOSED BOOK** exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave any unresolved integrals or summations in your answer. Other than that, however, you should not simplify explicit numerical expressions.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must **SHOW YOUR WORK** to get full credit.

Name: _____

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Linearity, Shift-Invariance, Causality, Stability

1. **Linearity:** $x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$
2. **Shift-Invariance:** $x_1[n - n_0] \rightarrow y_1[n - n_0]$
3. **Causality:** $h[n] = 0$ for $n < 0$
4. **Stability:** $|x[n]| < M_1 \Rightarrow |y[n]| < M_2$

Convolution

$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Sampling and Interpolation:

$$\begin{aligned}x[n] &= x\left(t = \frac{n}{F_s}\right) \\f_a &= \min(f \bmod F_s, -f \bmod F_s) \\z_a &= \begin{cases} z & f \bmod F_s < -f \bmod F_s \\ z^* & f \bmod F_s > -f \bmod F_s \end{cases} \\y(t) &= \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)\end{aligned}$$

Fourier Series

$$\begin{aligned}\text{Analysis: } X_k &= \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j2\pi kt/T_0} dt \\ \text{Synthesis: } x(t) &= \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}\end{aligned}$$

Discrete Fourier Transform

$$\begin{aligned}\text{Analysis: } X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \\ \text{Synthesis: } x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \\ \text{Circular Convolution: } x[n] \circledast h[n] &= \sum_{m=0}^{N-1} h[m]x[(n-m)_N]\end{aligned}$$

Frequency Response

$$\begin{aligned}H(\omega) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ h[n] * \cos(\omega n) &= |H(\omega)| \cos(\omega n + \angle H(\omega))\end{aligned}$$

1. (26 points) A system has input $x[n]$, and output $y[n] = 4x[n - 7] + \cos\left(\frac{\pi n}{5}\right)$.
(a) Is this system linear? Prove your answer.

Solution:

$$\begin{aligned}x_1[n] &\rightarrow y_1[n] = 4x_1[n - 7] + \cos\left(\frac{\pi n}{5}\right) \\x_2[n] &\rightarrow y_2[n] = 4x_2[n - 7] + \cos\left(\frac{\pi n}{5}\right) \\x_1[n] + x_2[n] &\rightarrow y_2[n] = 4x_1[n - 7] + 4x_2[n - 7] + \cos\left(\frac{\pi n}{5}\right) \\&= y_1[n] + y_2[n]\end{aligned}$$

Therefore it is not linear.

- (b) Is this system time-invariant? Prove your answer.

Solution:

$$\begin{aligned}x_1[n] &\rightarrow y_1[n] = 4x_1[n - 7] + \cos\left(\frac{\pi n}{5}\right) \\x_1[n - n_0] &\rightarrow y_2[n] = 4x_1[n - n_0 - 7] + \cos\left(\frac{\pi n}{5}\right) \\&= y_1[n - n_0]\end{aligned}$$

Therefore it is not time-invariant.

2. (26 points) The signal $x(t) = \sum_{k=-4}^4 15e^{j2000\pi kt}$ is sampled at 10,000 samples/second to produce $x[n]$, then processed by the system $y[n] = x[n-1] + x[n-5]$. The signal $y[n]$ is then interpolated using piece-wise linear interpolation, at 10,000 samples/second, to produce $y(t)$. The signal $y(t)$ is given by

$$y(t) = z(t) + \sum_{k=-4}^4 Y_k e^{j2000\pi kt}$$

where $z(t)$ contains some aliasing signal components at frequencies above 5000Hz.

- (a) At what frequencies does the signal $z(t)$ have nonzero energy?

Solution: Aliasing components include $mF_s \pm f$ for every f in the input signal, thus the aliasing components f_a include

$$f_a \in \{10000m \pm 1000k \text{ for } -4 \leq k \leq 4, m \neq 0\}$$

- (b) Give an expression for Y_k in terms of k for $-4 \leq k \leq 4$. Your answer should contain no variables except k .

Solution:

$$H(\omega) = 2 \cos(2\omega) e^{-3j\omega}$$

$$Y_k = 30 \cos(2k\omega_0) e^{-3jk\omega_0}$$

where $\omega_0 = \frac{2\pi F_0}{F_s} = \frac{2000\pi}{10,000} = 0.2\pi$, so

$$Y_k = 30 \cos(0.4\pi k) e^{-0.6j\pi k}$$

3. (24 points) $x(t)$ is a periodic signal with period T_0 . $x(t) = 1$ for $0 < t < 0.1T_0$ and for $0.2T_0 < t < 0.3T_0$, and it is zero elsewhere. Find $|X_k|$ and $\angle X_k$ for both $k = 0$ and $k \neq 0$.

Solution: For $k = 0$, we have

$$\begin{aligned} X_0 &= \frac{1}{T_0} \int_0^{T_0} x(t) dt \\ &= 0.2 \end{aligned}$$

For $k \neq 0$, we have

$$\begin{aligned} X_k &= \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j \frac{2\pi k t}{T_0}} dt \\ &= \frac{1}{T_0} \int_0^{0.1T_0} e^{-j \frac{2\pi k t}{T_0}} dt + \frac{1}{T_0} \int_{0.2T_0}^{0.3T_0} e^{-j \frac{2\pi k t}{T_0}} dt \\ &= \frac{1}{-j2k\pi} (e^{-j0.6k\pi} - e^{-j0.4k\pi} + e^{-j0.2k\pi} - 1) \\ &= j e^{-j0.3k\pi} \frac{1}{2k\pi} (e^{j0.3k\pi} - e^{-j0.1k\pi} + e^{-j0.1k\pi} - e^{-j0.3k\pi}) \\ &= j e^{-j0.3k\pi} \frac{1}{k\pi} (\cos(0.3k\pi) - \cos(0.1k\pi)) \\ |X_k| &= \frac{1}{k\pi} |\cos(0.3k\pi) - \cos(0.1k\pi)| \\ \angle X_k &= \begin{cases} -0.3k\pi + \frac{\pi}{2} & \cos(0.3k\pi) + \cos(0.1k\pi) > 0 \\ -0.3k\pi - \frac{\pi}{2} & \cos(0.3k\pi) + \cos(0.1k\pi) < 0 \end{cases} \end{aligned}$$

4. (24 points) Consider these two signals:

$$x[n] = \begin{cases} 1 & 0 \leq n < 200 \\ 0 & \text{otherwise} \end{cases}, \quad h[n] = \begin{cases} 1 & 0 \leq n < 100 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $X[k]$, $H[k]$, and $Y[k]$ are the 256-point discrete Fourier transforms of $x[n]$, $h[n]$, and $y[n]$, respectively, where $Y[k] = H[k]X[k]$. What is $y[n]$? You may sketch your answer, but be sure all numerical values are clearly labeled.

Solution: $y[n] = h[n] \otimes x[n]$ is a temporally aliased copy of $h[n] * x[n]$, which is

$$h[n] * x[n] = \begin{cases} n + 1 & 0 \leq n < 100 \\ 100 & 100 \leq n < 200 \\ 299 - n & 200 \leq n \leq 298 \\ 0 & \text{otherwise} \end{cases}$$

Temporally aliasing with a length of $N = 256$ will wrap around the last $298 - 256 = 42$ samples, giving

$$y[n] = \begin{cases} 44 & 0 \leq n \leq 43 \\ n + 1 & 44 \leq n < 100 \\ 100 & 100 \leq n < 200 \\ 299 - n & 200 \leq n \leq 255 \end{cases}$$

The definition we've used in class specifies that $x[n] = x[n - 256]$ outside the range shown above. The definitions used in most other classes would specify $x[n] = 0$ outside the range shown above. Either answer is acceptable.

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