Final Exam Review

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ECE 401: Signal and Image Analysis

Outline

Final Exam: General Structure

- About twice as long as a midterm (i.e., 8-10 problems with 1-3 parts each)
- You'll have 3 hours for the exam
- The usual rules: no calculators or computers, two sheets of handwritten notes, you will have two pages of formulas provided on the exam, published by the Friday before the exam.

Final Exam: Topics Covered

- 17%: Material from exam 1 (phasors, Fourier series)
- 17%: Material from exam 2 (LSI systems, DTFT)
- 66%: Material from the last third of the course (DFT, Z transform)

Material from the last third of the course

- DFT & Window Design
- Circular Convolution
- Z Transform & Inverse Z Transform
- Notch Filters & Second-Order IIR

Outline

DFT and Inverse DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

DFT of a Cosine

$$x[n] = \cos(\omega_0 n) w[n] \quad \leftrightarrow \quad X(\omega_k) = \frac{1}{2} W(\omega_k - \omega_0) + \frac{1}{2} W(\omega_k + \omega_0)$$

where $W(\omega)$ is the transform of $w[n]$. For example, if $w[n]$ is a rectangular window, then

$$W(\omega) = e^{-j\omega \frac{N-1}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Properties of the DFT

- The DFT is periodic in frequency:

$$X[k+N] = X[k]$$

- The inverse DFT is periodic in time: if $x[n]$ is the inverse DFT of $X[k]$, then

$$x[n+N] = x[n]$$

- Linearity:

$$ax_1[n] + bx_2[n] \quad \leftrightarrow \quad aX_1[k] + bX_2[k]$$

- Samples of the DTFT: if $x[n]$ is finite in time, with length $\leq N$, then

$$X[k] = X(\omega_k), \quad \omega_k = \frac{2\pi k}{N}$$

Properties of the DFT

- Conjugate symmetric:

$$X[k] = X^*[-k] = X^*[N-k]$$

- Frequency shift:

$$w[n]e^{j\frac{2\pi k_0 n}{N}} \quad \leftrightarrow \quad W[k-k_0]$$

- Circular time shift:

$$x\left[\langle n-n_0\rangle_N\right] \quad \leftrightarrow \quad e^{j\frac{2\pi k n_0}{N}} X[k]$$

Outline

DFT is actually a Fourier Series

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}$$

Circular Convolution

$$
\begin{aligned}
Y[k] &= H[k]X[k] \\
y[n] &= h[n] \circledast x[n] \\
&= \sum_{m=0}^{N-1} h\left[m\right] x\left[\langle n-m\rangle_N\right] \\
&= \sum_{m=0}^{N-1} x\left[m\right] h\left[\langle n-m\rangle_N\right]
\end{aligned}
$$

Outline

Z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

System Function

$$y[n] = 0.2x[n+3] + 0.3x[n+2] + 0.5x[n+1] - 0.5x[n-1] - 0.3x[n-2] - 0.2x[n-3]$$

$$H(z) = \frac{Y(z)}{X(z)} = 0.2z^3 + 0.3z^2 + 0.5z^1 - 0.5z^{-1} - 0.3z^{-2} - 0.2z^{-3}$$

The Zeros of $H(z)$

- The roots, z_1 and z_2, are the values of z for which $H(z) = 0$.
- But what does that mean? We know that for $z = e^{j\omega}$, $H(z)$ is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

but the roots do not have unit magnitude:

$$z_1 = 1 + j = \sqrt{2}e^{j\pi/4}$$
$$z_2 = 1 - j = \sqrt{2}e^{-j\pi/4}$$

- What it means is that, when $\omega = \frac{\pi}{4}$ (so $z = e^{j\pi/4}$), then $|H(\omega)|$ is as close to a zero as it can possibly get. So at that frequency, $|H(\omega)|$ is as low as it can get.

Outline

General form of an FIR filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

This filter has an impulse response ($h[n]$) that is $M+1$ samples long.

- The b_k's are called **feedforward** coefficients, because they feed $x[n]$ forward into $y[n]$.

General form of an IIR filter

$$\sum_{\ell=0}^{N} a_\ell y[n-\ell] = \sum_{k=0}^{M} b_k x[n-k]$$

- The a_ℓ's are caled **feedback** coefficients, because they feed $y[n]$ back into itself.

Transfer Function of a First-Order Filter

We can find the transfer function by taking the Z-transform of each term in this equation equation:

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$
$$Y(z) = X(z) + bz^{-1}X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + bz^{-1}}{1 - az^{-1}}$$

The Pole and Zero of $H(z)$

- The pole, $z = a$, and zero, $z = -b$, are the values of z for which $H(z) = \infty$ and $H(z) = 0$, respectively.
- But what does that mean? We know that for $z = e^{j\omega}$, $H(z)$ is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

  but the pole and zero do not normally have unit magnitude.
- What it means is that:
  - When $\omega = \angle(-b)$, then $|H(\omega)|$ is as close to a zero as it can possibly get, so at that that frequency, $|H(\omega)|$ is as low as it can get.
  - When $\omega = \angle a$, then $|H(\omega)|$ is as close to a pole as it can possibly get, so at that that frequency, $|H(\omega)|$ is as high as it can get.

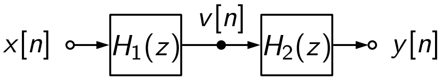
Causality and Stability

- A filter is **causal** if and only if the output, $y[n]$, depends only an **current and past** values of the input, $x[n], x[n-1], x[n-2], \ldots$.
- A filter is **stable** if and only if **every** finite-valued input generates a finite-valued output. A causal first-order IIR filter is stable if and only if $|a| < 1$.

Outline

Series Combination

The series combination of two systems looks like this:



This means that

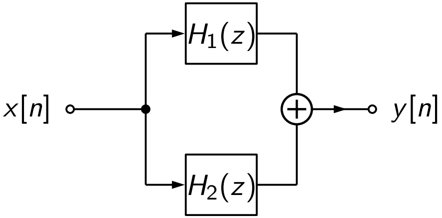
$$Y(z) = H_2(z)V(z) = H_2(z)H_1(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z)$$

Parallel Combination

Parallel combination of two systems looks like this:



This means that

$$Y(z) = H_1(z)X(z) + H_2(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) + H_2(z)$$

How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as...

- **denominator terms**, each with this form:

$$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a **numerator** term, like this one:

$$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n-k].$$

Step #1: Numerator Terms

In general, if

$$G(z)=\frac{1}{A(z)}$$

for any polynomial $A(z)$, and

$$H(z)=\frac{\sum_{k=0}^{M}b_k z^{-k}}{A(z)}$$

then

$$h[n]=b_0g[n]+b_1g[n-1]+\cdots+b_Mg[n-M]$$

Step #2: Partial Fraction Expansion

Partial fraction expansion works like this:

1. Factor $A(z)$:

$$G(z) = \frac{1}{\prod_{\ell=1}^{N}\left(1 - p_\ell z^{-1}\right)}$$

2. Assume that $G(z)$ is the result of a parallel system combination:

$$G(z) = \frac{C_1}{1 - p_1 z^{-1}} + \frac{C_2}{1 - p_2 z^{-1}} + \cdots$$

3. Find the constants, C_ℓ, that make the equation true. Such constants always exist, as long as none of the roots are repeated ($p_k \neq p_\ell$ for $k \neq \ell$).

Outline

How to Implement a Notch Filter

To implement a notch filter at frequency ω_c radians/sample, with a bandwidth of $-\ln(a)$ radians/sample, you implement the difference equation:

$$y[n] = x[n] - 2\cos(\omega_c)x[n-1] + x[n-2] + 2a\cos(\omega_c)y[n-1] - a^2 y[n-2]$$

which gives you the notch filter

$$H(z) = \frac{(1 - r_1 z^{-1})(1 - r_1^* z^{-1})}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})}$$

with the magnitude response:

$$|H(\omega)| = \begin{cases} 0 & \omega_c \\ \frac{1}{\sqrt{2}} & \omega_c \pm \ln(a) \\ \approx 1 & \omega < \omega + \ln(a) \text{ or } \omega > \omega - \ln(a) \end{cases}$$

Outline

A General Second-Order All-Pole Filter

Let's construct a general second-order all-pole filter (leaving out the zeros; they're easy to add later).

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})} = \frac{1}{1 - (p_1 + p_1^*)z^{-1} + p_1 p_1^* z^{-2}}$$

The difference equation that implements this filter is

$$Y(z) = X(z) + (p_1 + p_1^*)z^{-1}Y(z) - p_1 p_1^* z^{-2} Y(z)$$

Which converts to

$$y[n] = x[n] + 2\Re(p_1)y[n-1] - |p_1|^2 y[n-2]$$

Understanding the Impulse Response of a Second-Order IIR

In order to **understand** the impulse response, maybe we should invent some more variables. Let's say that

$$p_1 = e^{-\sigma_1 + j\omega_1}, \quad p_1^* = e^{-\sigma_1 - j\omega_1}$$

where σ_1 is the half-bandwidth of the pole, and ω_1 is its center frequency. The partial fraction expansion gave us the constant

$$C_1 = \frac{p_1}{p_1 - p_1^*} = \frac{p_1}{e^{-\sigma_1}\left(e^{j\omega_1} - e^{-j\omega_1}\right)} = \frac{e^{j\omega_1}}{2j\sin(\omega_1)}$$

Therefore

$$h[n] = \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin(\omega_1(n+1)) u[n]$$

Example: Ideal Resonator

Putting $p_1 = e^{j\omega_1}$ into the general form, we find that the impulse response of this filter is

$$h[n] = \frac{1}{\sin(\omega_1)} \sin(\omega_1(n+1))u[n]$$

This is called an "ideal resonator" because it keeps ringing forever.

Bandwidth

There are three frequencies that really matter:

1. Right at the pole, at $\omega = \omega_1$, we have

$$|e^{j\omega} - p_1| \approx \sigma_1$$

2. At $\pm$ half a bandwidth, $\omega = \omega_1 \pm \sigma_1$, we have

$$|e^{j\omega} - p_1| \approx |-\sigma_1 \mp j\sigma_1| = \sigma_1\sqrt{2}$$

3dB Bandwidth

- The 3dB bandwidth of an all-pole filter is the width of the peak, measured at a level $1/\sqrt{2}$ relative to its peak.
- σ_1 is half the bandwidth.

Outline

Summary

- DFT & Window Design
- Circular Convolution
- Z Transform & Inverse Z Transform
- Notch Filters & Second-Order IIR