

# Lecture 28: Z Transform

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ECE 401: Signal and Image Analysis

- 1 Review: DTFT
- 2 Difference Equations
- 3 Every Signal is a Weighted Sum of Impulses
- 4 Z Transform
- 5 Finding the Zeros of  $H(z)$
- 6 Finding the Poles and Zeros of  $H(z)$
- 7 Summary

# Outline

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# DTFT

The DTFT (discrete time Fourier transform) of any signal is  $X(\omega)$ , given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$

$$g[n] = \delta[n - n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$



# Ideal Filters

Ideal Lowpass Filter:

$$H_{LPF}(\omega) = \begin{cases} 1 & |\omega| \leq \omega_L, \\ 0 & \omega_c < |\omega| \leq \pi. \end{cases} \quad \leftrightarrow \quad h_{LPF}[m] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n)$$

# Ideal Filters

Ideal Bandpass Filter:

$$H_{BPF}(\omega) = H_{LPF}(\omega|\omega_2) - H_{LPF}(\omega|\omega_1)$$

$$\leftrightarrow h_{BPF}[n] = \frac{\omega_2}{\pi} \text{sinc}(\omega_2 n) - \frac{\omega_1}{\pi} \text{sinc}(\omega_1 n)$$

# Ideal Filters

Delayed Ideal Highpass Filter:

$$H_{HPF}(\omega) = e^{-j\omega n_0} (1 - H_{LPF}(\omega))$$

$\leftrightarrow$

$$h_{HPF}[n] = \begin{cases} \delta[n - n_0] - \frac{\omega_c}{\pi} \text{sinc}(\omega_c(n - n_0)) & n_0 = \text{integer} \\ \text{sinc}(\pi(n - n_0)) - \frac{\omega_c}{\pi} \text{sinc}(\omega_c(n - n_0)) & \text{otherwise} \end{cases}$$

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# Linearity and Time-Shift Properties

The linearity property of the DTFT says that

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega).$$

The time-shift property says that

$$z[n] = x[n - n_0] \leftrightarrow Z(\omega) = e^{-j\omega n_0} X(\omega)$$

# Difference Equation

A **difference equation** is an equation in terms of time-shifted values of multiple signals. For example,

$$y[n] = x[n] - 2x[n - 1] + 2x[n - 2]$$

By combining the linearity and time-shift properties of the DTFT, we can translate the whole difference equation into the frequency domain as

$$Y(\omega) = X(\omega) - 2e^{-j\omega} X(\omega) + 2e^{-2j\omega} X(\omega)$$

# Impulse Response

A difference equation implements a discrete-time filter. Therefore, it has an **impulse response**. You can find the impulse response by just putting in an impulse,  $x[n] = \delta[n]$ , and seeing how it responds. Whatever value of  $y[n]$  that comes out of the filter is the impulse response:

$$h[n] = \delta[n] - 2\delta[n - 1] + 2\delta[n - 2] = \begin{cases} 1 & n = 0 \\ -2 & n = 1 \\ 2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

If you wanted to use `np.convolve` to implement this filter, you now know what impulse response to use.

$$\begin{aligned} y[n] &= x[n] - 2x[n - 1] + 2x[n - 2] \\ &= \sum_{m=0}^2 h[m]x[n - m] \end{aligned}$$

# Frequency Response

The **frequency response** of a filter is the function  $H(\omega)$  such that  $Y(\omega) = H(\omega)X(\omega)$ , or in other words,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

We can get this from the DTFT of the difference equation:

$$Y(\omega) = (1 - 2e^{-j\omega} + 2e^{-2j\omega}) X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = (1 - 2e^{-j\omega} + 2e^{-2j\omega})$$



# Frequency Response is DTFT of the Impulse Response

The frequency response is the DTFT of the impulse response.

$$h[n] = \delta[n] - 2\delta[n - 1] + 2\delta[n - 2]$$

$$H(\omega) = 1 - 2e^{-j\omega} + 2e^{-2j\omega}$$

It sounds like an accident, that frequency response is DTFT of the impulse response. But actually, it's because the method for computing frequency response and the method for computing impulse response do the same two steps, in the opposite order (see next slide).

## How to compute the frequency response

- 1 Take the DTFT of every term, so that  $ax[n - n_0]$  is converted to  $ae^{-j\omega n_0}X(\omega)$ .
- 2 Divide by  $X(\omega)$ .

## How to compute the DTFT of the impulse response

- 1 Replace  $x[n]$  by  $\delta[n]$ , so that  $ax[n - n_0]$  is converted to  $a\delta[n - n_0]$ .
- 2 Take the DTFT, so that  $a\delta[n - n_0]$  becomes  $ae^{-j\omega n_0}$ .

## How to compute the frequency response

- 1 Take the DTFT of every term:

$$Y(\omega) = X(\omega) - 2e^{-j\omega}X(\omega) + 2e^{-2j\omega}X(\omega)$$

- 2 Divide by  $X(\omega)$ .

$$H(\omega) = 1 - 2e^{-j\omega} + 2e^{-2j\omega}$$

## How to compute the DTFT of the impulse response

- 1 Replace  $x[n]$  by  $\delta[n]$ :

$$h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2]$$

- 2 Take the DTFT:

$$H(\omega) = 1 - 2e^{-j\omega} + 2e^{-2j\omega}$$

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# Definition of the DTFT

The definition of the DTFT is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- We viewed this, before, as computing the phasor  $X(\omega)$  using the orthogonality principle: multiply  $x[n]$  by a pure tone at the corresponding frequency, and sum over all time.
- But there's another useful way to think about this.

# A Signal is a Weighted Sum of Impulses

Try writing the DTFT as

$$X(\omega) = \dots + x[-1]e^{j\omega} + x[0] + x[1]e^{-j\omega} + \dots$$

This looks like the DTFT of a difference equation. The inverse DTFT would be

$$x[n] = \dots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \dots$$

# A Signal is a Weighted Sum of Impulses

# A Signal is a Weighted Sum of Impulses

So we can use the DTFT formula,  $X(\omega) = \sum x[n]e^{-j\omega n}$ , to inspire us to think about  $x[n]$  as just a weighted sum of impulses:

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \leftrightarrow x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n - m]$$



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## Z: a Frequency Variable for Time Shifts

If we're going to be working a lot with delays, instead of pure tones, then it helps to change our frequency variable. Until now, we've been working in terms of  $\omega$ , the frequency of the pure tone:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

A unit delay,  $\delta[n - 1]$ , has the DTFT  $e^{-j\omega}$ . Just to reduce the notation a little, let's define the basic unit to be  $z = e^{j\omega}$ , which is the transform of  $\delta[n + 1]$  (a unit advance). Then we get:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

# Main Use of Z Transform: Difference Equations

The main purpose of the Z transform, for now, is just so that we have less to write. Instead of transforming

$$y[n] = x[n] - 2x[n-1] + 2x[n-2]$$

to get

$$Y(\omega) = (1 - 2e^{-j\omega} + 2e^{-2j\omega}) X(\omega)$$

Now we can just write

$$Y(z) = (1 - 2z^{-1} + 2z^{-2}) X(z)$$

# Main Use of Z Transform: Difference Equations

The longer the difference equation, the more you will appreciate writing  $z$  instead of  $e^{j\omega}$ .

$$y[n] = 0.2x[n + 3] + 0.3x[n + 2] + 0.5x[n + 1] \\ - 0.5x[n - 1] - 0.3x[n - 2] - 0.2x[n - 2]$$

$$H(z) = \frac{Y(z)}{X(z)} = 0.2z^3 + 0.3z^2 + 0.5z^1 - 0.5z^{-1} - 0.3z^{-2} - 0.2z^{-3}$$

# A Signal is a Weighted Sum of Impulses

Remember that a signal is just a weighted sum of impulses?

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

Since the Z-transform of  $\delta[n-m]$  is  $z^{-m}$ , we can transform the above difference equation to get

$$X(z) = \sum_{m=-\infty}^{\infty} x[m]z^{-m}.$$

It's like we've converted a weighted sum of impulses ( $x[n]$ ) into a polynomial in  $z$  ( $X(z)$ ).

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# Z Transform and Convolution

Here's a formula for convolution:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Since the Z-transform of  $x[n-m]$  is  $z^{-m}X(z)$ , we can transform the above difference equation to get

$$\begin{aligned} Y(z) &= \sum_{m=-\infty}^{\infty} h[m]z^{-m}X(z) \\ &= H(z)X(z) \end{aligned}$$

So we confirm that  $x[n] * h[n] \leftrightarrow H(z)X(z)$ .

# Treating $H(z)$ as a Polynomial

Suppose we have

$$h[n] = \delta[n] - 2\delta[n - 1] + 2\delta[n - 2]$$

- Is this a low-pass filter, a high-pass filter, or something else?
- $H(z)$  provides a new way of thinking about the frequency response: not an ideal LPF or HPF or BPF, but instead, something with particular **zeros** in the frequency domain.



# Treating $H(z)$ as a Polynomial

Here's the transfer function  $H(z)$ :

$$H(z) = 1 - 2z^{-1} + 2z^{-2}$$

Notice that we can factor that, just like any other polynomial:

$$H(z) = \frac{1}{z^2} (z^2 - 2z + 2)$$

Using the quadratic formula, we can find its roots:

$$z = \frac{2 \pm \sqrt{(2)^2 - 4 \times 2}}{2} = 1 \pm j$$

# Treating $H(z)$ as a Polynomial

We've discovered that  $H(z)$  can be written as a product of factors:

$$H(z) = 1 - 2z^{-1} + 2z^{-2} = \frac{1}{z^2}(z - z_1)(z - z_2),$$

where the roots of the polynomial are

$$z_1 = 1 + j = \sqrt{2}e^{j\pi/4}$$

$$z_2 = 1 - j = \sqrt{2}e^{-j\pi/4}$$

# The Zeros of $H(z)$

- The roots,  $z_1$  and  $z_2$ , are the values of  $z$  for which  $H(z) = 0$ .
- But what does that mean? We know that for  $z = e^{j\omega}$ ,  $H(z)$  is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

but the roots do not have unit magnitude:

$$z_1 = 1 + j = \sqrt{2}e^{j\pi/4}$$

$$z_2 = 1 - j = \sqrt{2}e^{-j\pi/4}$$

- What it means is that, when  $\omega = \frac{\pi}{4}$  (so  $z = e^{j\pi/4}$ ), then  $|H(\omega)|$  is as close to a zero as it can possibly get. So at that frequency,  $|H(\omega)|$  is as low as it can get.



DTFT  
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Difference Equations  
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Impulses  
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Z Transform  
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Zeros  
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Poles and Zeros  
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Summary  
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# Vectors in the Complex Plane

Suppose we write  $|H(z)|$  like this:

$$|H(z)| = \frac{1}{|z|^2} \times |z - z_1| \times |z - z_2| = |z - z_1| \times |z - z_2|$$

Now let's evaluate at  $z = e^{j\omega}$ :

$$|H(\omega)| = |e^{j\omega} - z_1| \times |e^{j\omega} - z_2|$$

What we've discovered is that  $|H(\omega)|$  is small when the vector distance  $|e^{j\omega} - z_1|$  is small, in other words, when  $z = e^{j\omega}$  is as close as possible to one of the zeros.

DTFT  
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Try the quiz!

Try the quiz!



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# First-Order Filter

Now, let's find the transfer function of a general first-order filter, including BOTH feedforward and feedback delays:

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$

where we'll assume that  $|a| < 1$ , so the filter is stable.

# Transfer Function of a First-Order Filter

We can find the transfer function by taking the Z-transform of each term in this equation:

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$
$$Y(z) = X(z) + bz^{-1}X(z) + az^{-1}Y(z),$$

which we can solve to get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + bz^{-1}}{1 - az^{-1}}.$$

# Treating $H(z)$ as a Ratio of Two Polynomials

Notice that  $H(z)$  is the ratio of two polynomials:

$$H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}} = \frac{z + b}{z - a}$$

- $z = -b$  is called the **zero** of  $H(z)$ , meaning that  $H(-b) = 0$ .
- $z = a$  is called the **pole** of  $H(z)$ , meaning that  $H(a) = \infty$

# The Pole and Zero of $H(z)$

- The pole,  $z = a$ , and zero,  $z = -b$ , are the values of  $z$  for which  $H(z) = \infty$  and  $H(z) = 0$ , respectively.
- But what does that mean? We know that for  $z = e^{j\omega}$ ,  $H(z)$  is just the frequency response:

$$H(\omega) = H(z)|_{z=e^{j\omega}}$$

but the pole and zero do not normally have unit magnitude.

- What it means is that:
  - When  $\omega = \angle(-b)$ , then  $|H(\omega)|$  is as close to a zero as it can possibly get, so at that that frequency,  $|H(\omega)|$  is as low as it can get.
  - When  $\omega = \angle a$ , then  $|H(\omega)|$  is as close to a pole as it can possibly get, so at that that frequency,  $|H(\omega)|$  is as high as it can get.

DTFT  
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Difference Equations  
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Summary  
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# Vectors in the Complex Plane

Suppose we write  $|H(z)|$  like this:

$$|H(z)| = \frac{|z + b|}{|z - a|}$$

Now let's evaluate at  $z = e^{j\omega}$ :

$$|H(\omega)| = \frac{|e^{j\omega} + b|}{|e^{j\omega} - a|}$$

What we've discovered is that  $|H(\omega)|$  is small when the vector distance  $|e^{j\omega} + b|$  is small, but LARGE when the vector distance  $|e^{j\omega} - a|$  is small.



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# Summary: Z Transform

- A **difference equation** is an equation in terms of time-shifted copies of  $x[n]$  and/or  $y[n]$ .
- We can find the frequency response  $H(\omega) = Y(\omega)/X(\omega)$  by taking the DTFT of each term of the difference equation. This will result in a lot of terms of the form  $e^{j\omega n_0}$  for various  $n_0$ .
- We have less to write if we use a new frequency variable,  $z = e^{j\omega}$ . This leads us to the Z transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

# Zeros of the Transfer Function

- The **transfer function**,  $H(z)$ , is a polynomial in  $z$ .
- The zeros of the transfer function are usually complex numbers,  $z_k$ .
- The frequency response,  $H(\omega) = H(z)|_{z=e^{j\omega}}$ , has a dip whenever  $\omega$  equals the phase of any of the zeros,  $\omega = \angle z_k$ .