# Lecture 27: IIR Filters

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ECE 401: Signal and Image Analysis

# Outline

# Review: Frequency Response

- **Tones in → Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos(\omega n + \angle H(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)| \cos(\omega n + \theta + \angle H(\omega))$$

- where the **Frequency Response** is given by

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

# Example: First Difference

$$y[n] = x[n] - x[n-1] = \sum_m h[m]x[n-m]$$

$$h[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H(\omega) = \sum_m h[m]e^{-j\omega m} = 1 - e^{-j\omega}$$

# Another way to think about first difference

$$y[n] = x[n] - x[n-1] \tag{1}$$

But remember the delay property of the DTFT:

$$x[n-k] \leftrightarrow e^{-j\omega k} X(\omega)$$

So we could take the DTFT of each term in Eq. (1) to get:

$$Y(\omega) = X(\omega) - e^{-j\omega} X(\omega) = \left(1 - e^{-j\omega}\right) X(\omega)$$

# Another way to think about first difference

$$Y(\omega) = X(\omega) - e^{-j\omega}X(\omega) = \left(1 - e^{-j\omega}\right)X(\omega)$$

But remember the convolution property of the DTFT:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

So we find that:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 - e^{-j\omega}$$

# Outline

# Autoregressive Filter

Today, instead of filters with this form:

$$y[n] = x[n] - x[n-1]$$

. . . we want to start studying filters of this form:

$$y[n] = x[n] - y[n-1]$$

- This is called **autoregressive**, meaning that $y[n]$ depends on past values (regressive) of itself (auto).
- This is also an **infinite impulse response** (IIR) filter, because the impulse response ($h[n]$) is infinitely long.

# Autoregressive Difference Equations

An **autoregressive** filter is one in which the output, $y[n]$, depends on past values of itself (**auto**=self, **regress**=go back). For example,

$$y[n] = x[n] + 0.3x[n-1] + 0.8y[n-1]$$

# Causal and Anti-Causal Filters

- If the outputs of a filter depend only on **current and past** values of the input, then the filter is said to be **causal**. An example is

$$y[n] = x[n] + 0.3x[n-1] + 0.8y[n-1]$$

- If the outputs depend only on **current and future** values of the input, the filter is said to be **anti-causal**, for example

$$y[n] = x[n] + 0.3x[n+1] + 0.8y[n+1]$$

- If the filter is neither causal nor anti-causal, we say it's "non-causal."
- Feedforward non-causal filters are easy to analyze, but when analyzing feedback, we will stick to causal filters.

# Autoregressive Difference Equations

We can find the frequency response by taking the DTFT of each term in the equation:

$$y[n] = x[n] + 0.3x[n-1] + 0.8y[n-1]$$
$$Y(\omega) = X(\omega) + 0.3e^{-j\omega}X(\omega) + 0.8e^{-j\omega}Y(z)$$

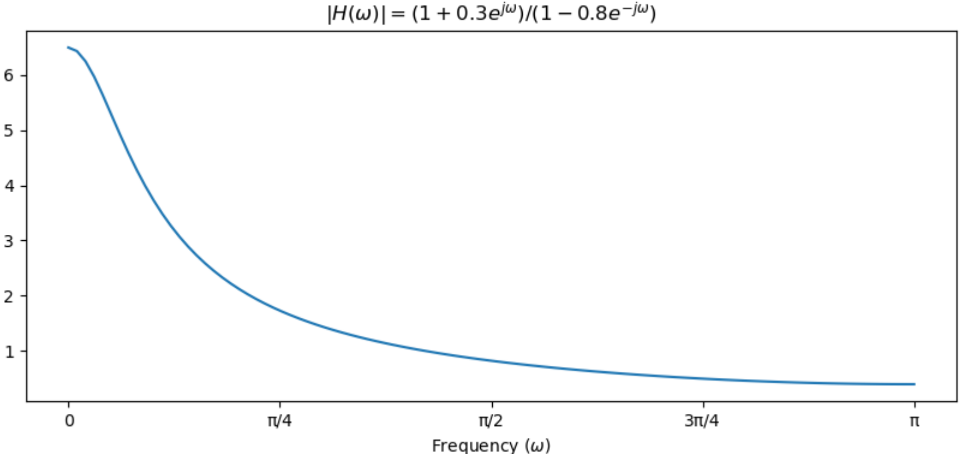
# Frequency Response

In order to find the frequency response, we need to solve for $H(\omega) = \frac{Y(\omega)}{X(\omega)}$.

$$\begin{aligned} Y(\omega) &= X(\omega) + 0.3e^{-j\omega}X(\omega) + 0.8e^{-j\omega}Y(\omega) \\ \left(1 - 0.8e^{-j\omega}\right) Y(\omega) &= X(\omega)(1 + 0.3e^{-j\omega}) \\ H(\omega) = \frac{Y(\omega)}{X(\omega)} &= \frac{1 + 0.3e^{-j\omega}}{1 - 0.8e^{-j\omega}} \end{aligned}$$

# Frequency Response

Here is the frequency response of this filter, plotted using
`np.abs((1+0.3*np.exp(-1j*omega))/(1-0.8*np.exp(-1j*omega)))`



# Outline

# Impulse Response of an Autoregressive Filter

One way to find the **impulse response** of an autoregressive filter is the same as for any other filter: feed in an impulse, $x[n] = \delta[n]$, and what comes out is the impulse response, $y[n] = h[n]$.

$$h[n] = \delta[n] + 0.3\delta[n-1] + 0.8h[n-1]$$

$$\begin{aligned}
h[n] &= 0, \quad n < 0 \\
h[0] &= \delta[0] = 1 \\
h[1] &= 0 + 0.3\delta[0] + 0.8h[0] = 1.1 \\
h[2] &= 0 + 0 + 0.8h[1] = 0.88 \\
h[3] &= 0 + 0 + 0.8h[2] = 0.704 \\
&\vdots \\
h[n] &= 1.1(0.8)^{n-1} \quad \text{if } n \geq 1 \\
&\vdots
\end{aligned}$$

# FIR vs. IIR Filters

- Most autoregressive filters are also **infinite impulse response (IIR)** filters, because $h[n]$ is infinitely long (never ends).
- A difference equation with only feedforward terms (like we saw in the last lecture) is always a **finite impulse response (FIR)** filter, because $h[n]$ has finite length.

# General form of an FIR filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_k h[k]x[n-k]$$

$$h[k] = \begin{cases} b_k & 0 \le k \le M \\ 0 & \text{otherwise} \end{cases}$$

This filter has an impulse response ($h[n]$) that is $M+1$ samples long.

- The $b_k$'s are called **feedforward** coefficients, because they feed $x[n]$ forward into $y[n]$.

# General form of an IIR filter

$$\sum_{\ell=0}^{N} a_\ell y[n-\ell] = \sum_{k=0}^{M} b_k x[n-k]$$

- The $a_\ell$'s are caled **feedback** coefficients, because they feed $y[n]$ back into itself.
- Can we find $h[n]$ in terms of $a_\ell$ and $b_k$?

# Outline

# First-Order Feedback-Only Filter

Let's find the general form of $h[n]$, for the simplest possible autoregressive filter: a filter with one feedback term, and no feedforward terms, like this:

$$y[n] = x[n] + ay[n-1],$$

where $a$ is any constant (positive, negative, real, or complex).

# Impulse Response of a First-Order Filter

We can find the impulse response by putting in $x[n] = \delta[n]$, and getting out $y[n] = h[n]$:

$$h[n] = \delta[n] + ah[n-1].$$

Recursive computation gives

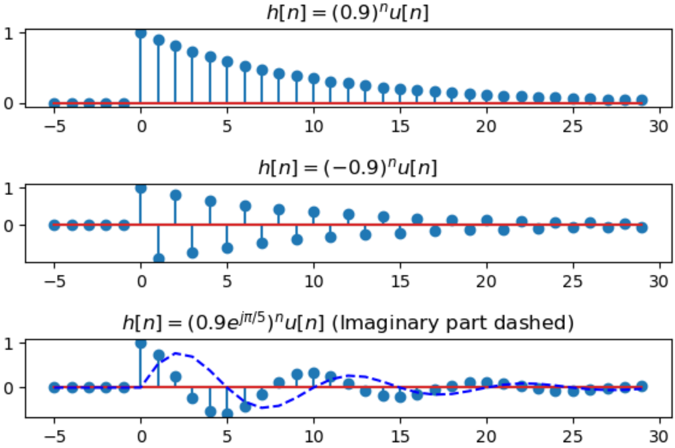
$$\begin{aligned} h[0] &= 1 \\ h[1] &= a \\ h[2] &= a^2 \\ &\vdots \\ h[n] &= a^n u[n] \end{aligned}$$

where we use the notation $u[n]$ to mean the "unit step function,"

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

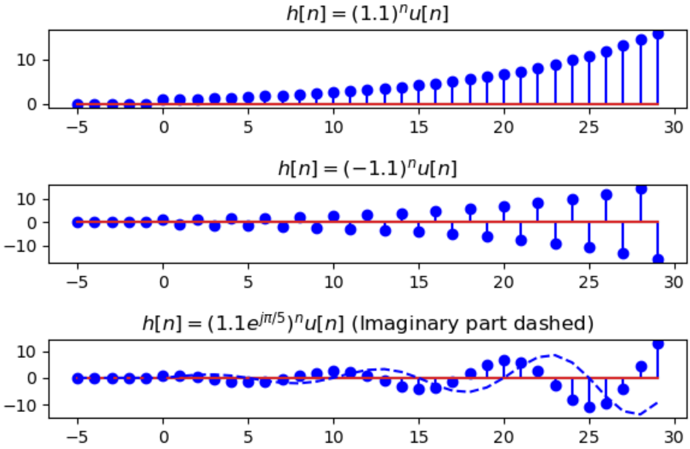
# Impulse Response of Stable First-Order Filters

The coefficient, $a$, can be positive, negative, or even complex. If $a$ is complex, then $h[n]$ is also complex-valued.



# Impulse Response of Unstable First-Order Filters

If $|a| > 1$, then the impulse response grows exponentially. If $|a| = 1$, then the impulse response never dies away. In either case, we say the filter is "unstable."



# Instability

- A **stable** filter is one that always generates finite outputs ($|y[n]|$ finite) for every possible finite input ($|x[n]|$ finite).
- An **unstable** filter is one that, at least sometimes, generates infinite outputs, even if the input is finite.
- A first-order IIR filter is stable if and only if $|a| < 1$.

# Frequency Response of a First-Order Filter

If the filter is stable ($|a| < 1$), then we can find the frequency response by taking the DTFT of $h[n]$:

$$h[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n \\ &= \begin{cases} \frac{1}{1-ae^{-j\omega}} & |a| < 1 \\ \infty & |a| \geq 1 \end{cases} \end{aligned}$$

# Frequency Response of a First-Order Filter

If the filter is stable ($|a| < 1$), then we can also find the frequency response by taking the DTFT of each term in the filter equation:

$$y[n] = x[n] - ay[n-1],$$

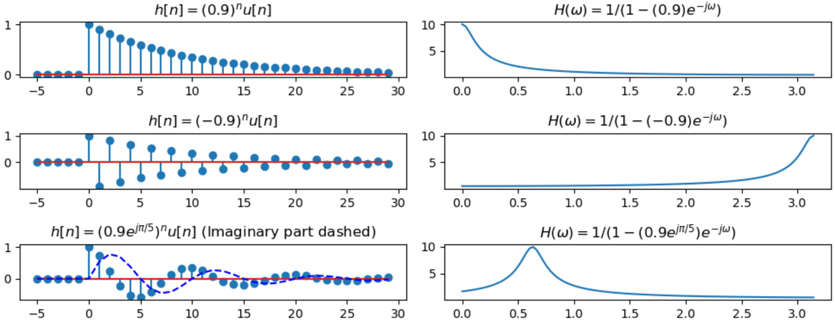
$$Y(\omega) = X(\omega) - ae^{-j\omega}Y(\omega),$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 - ae^{-j\omega}}$$

That **looks** like it works even if $|a| \geq 1$, but it's a lie. If $|a| \geq 1$, then when you put a pure tone as input, you might get $y[n] = \infty$ as output instead of $y[n] = H(\omega)x[n]$.

# Frequency Response of a First-Order Filter

$$H(\omega) = \frac{1}{1 - ae^{-j\omega}} \quad \text{iff } |a| < 1$$



# Try the quiz!

Try the quiz!

# Outline

# First-Order Filter

Now, let's find the frequency response of a general first-order filter, including BOTH feedforward and feedback delays:

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$

where we'll assume that $|a| < 1$, so the filter is stable.

# Frequency Response of a First-Order Filter

If $|a| < 1$, we can find the frequency response by taking the DTFT of each term in this equation:

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$
$$Y(\omega) = X(\omega) + be^{-j\omega}X(\omega) + ae^{-j\omega}Y(\omega),$$

which we can solve to get

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 + be^{-j\omega}}{1 - ae^{-j\omega}}.$$

# Treating $H(\omega)$ as a Ratio of Two Polynomials

Notice that $H(\omega)$ is the ratio of two polynomials:

$$H(\omega) = \frac{1 + be^{-j\omega}}{1 - ae^{-j\omega}} = \frac{e^{j\omega} + b}{e^{j\omega} - a}$$

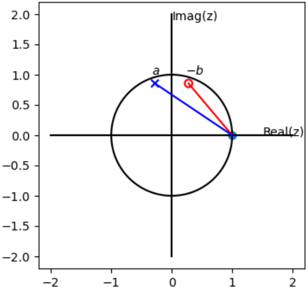
- $e^{j\omega} = -b$ is called the **zero** of $H(\omega)$, meaning that, if $|b| = 1$, then $H(\omega) = 0$ at $\omega = \angle(-b)$.
- $e^{j\omega} = a$ is called the **pole** of $H(\omega)$, meaning that, in the limit as $|a| \rightarrow 1$, $H(\angle a) \rightarrow \infty$.

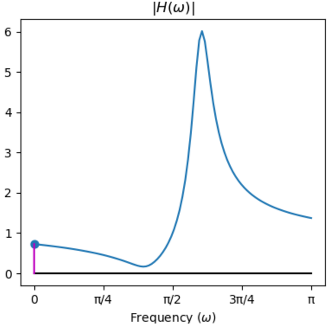
# Vectors in the Complex Plane

Suppose we write $|H(\omega)|$ like this:

$$|H(\omega)| = \frac{|e^{j\omega} + b|}{|e^{j\omega} - a|}$$

What we've discovered is that $|H(\omega)|$ is small when the vector distance $|e^{j\omega} + b|$ is small, but LARGE when the vector distance $|e^{j\omega} - a|$ is small.


Imag(z)
Real(z)
a
−b
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2


|H(ω)|
6
5
4
3
2
1
0
0
π/4
π/2
3π/4
π
Frequency (ω)

# Why This is Useful

Now we have another way of thinking about frequency response.

- Instead of just LPF, HPF, or BPF, we can design a filter to have zeros at particular frequencies, $\angle(-b)$, AND to have poles at particular frequencies, $\angle a$,
- The magnitude $|H(\omega)|$ is $|e^{j\omega}+b|/|e^{j\omega}-a|$.
- Using this trick, we can design filters that have much more subtle frequency responses than just an ideal LPF, BPF, or HPF.

# Outline

# Summary: Autoregressive Filter

- An **autoregressive filter** is a filter whose current output, $y[n]$, depends on past values of the output.
- An autoregressive filter is usually **infinite impulse response (IIR)**, because $h[n]$ has infinite length.
- A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).
- The first-order, feedback-only autoregressive filter has this impulse response and frequency response:

$$h[n] = a^n u[n] \leftrightarrow H(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

# Summary: Poles and Zeros

A first-order autoregressive filter,

$$y[n] = x[n] + bx[n-1] + ay[n-1],$$

has the impulse response and frequency response:

$$h[n] = a^n u[n] + ba^{n-1} u[n-1] \leftrightarrow H(\omega) = \frac{1 + be^{-j\omega}}{1 - ae^{-j\omega}},$$

where $a$ is called the **pole** of the filter, and $-b$ is called its **zero**.