Lecture 26: DTFT of a Sinusoid

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ECE 401: Signal and Image Analysis

Outline

Review: DFT, DTFT, and Fourier Series

Magnitude-summable signals have a DTFT:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Periodic signals have a Fourier series:

$$X_k = \frac{1}{N} \sum_{n=1}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}, \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}}$$

Finite-length or periodic signals have a DFT:

$$X[k] = \sum_{n=1}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}, \quad \Leftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

Review: DFT of a Sinusoid

To find the DFT of a sinusoid, we use the frequency-shift property of the DFT:

$$x[n] = \cos(\omega_0 n)w[n] = \left(\frac{1}{2}w[n]e^{j\omega_0 n} + \frac{1}{2}w[n]e^{-j\omega_0 n}\right)$$

$$\leftrightarrow$$

$$X[k] = \frac{1}{2}W\left(\frac{2\pi k}{N} - \omega_0\right) + \frac{1}{2}W\left(\frac{2\pi k}{N} + \omega_0\right)$$

where $W(\omega)$ is the DTFT of the window.

Today's Questions

Today's questions are:

1. Can we use the frequency-shift property to find the DTFT of a windowed sinusoid?
2. Can we use something like that to find the DTFT of a non-windowed, infinite length sinusoid?

Outline

DTFT of a Windowed Sinusoid

First, let's find the DTFT of a windowed sinusoid. This is easy; it's the same as the DFT. Since

$$x[n] = \cos(\omega_0 n)w[n] = \left(\frac{1}{2}w[n]e^{j\omega_0 n} + \frac{1}{2}w[n]e^{-j\omega_0 n}\right)$$

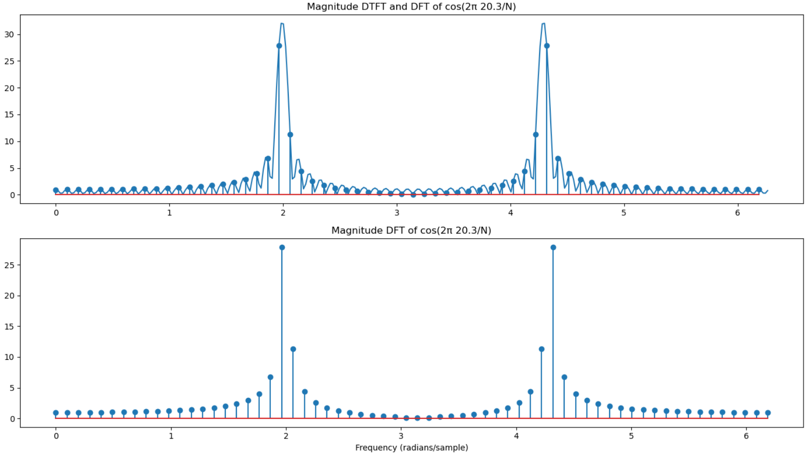
We can just use the frequency-shift property of the DTFT to get

$$X(\omega) = \frac{1}{2}W\left(\omega - \omega_0\right) + \frac{1}{2}W\left(\omega + \omega_0\right)$$

DFT of a Cosine

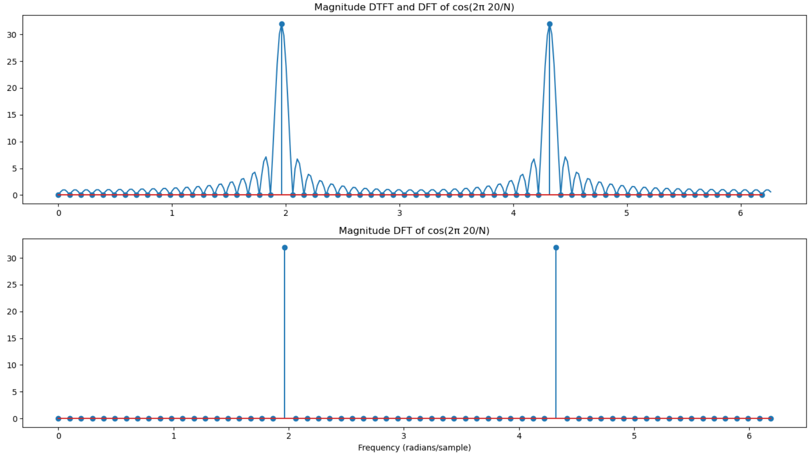
Here are the DTFT and DFT of

$$x[n] = \cos\left(\frac{2\pi 20.3}{N} n\right) w[n]$$



DFT of a Cosine

Here are the DTFT and DFT of a cosine at a frequency that's a multiple of $2\pi k/N$.



Outline

DTFT of a Non-Windowed Sinusoid

- How about $x[n] = \cos(\omega_0 n)$, with no windows? Does it have a DTFT?
- It's not magnitude-summable!

$$\sum_{n=-\infty}^{\infty} |x[n]| = \infty$$

  Therefore, there's no guarantee that it has a valid DTFT.
- In fact, we will need to make up some new math in order to find the DTFT of this signal.

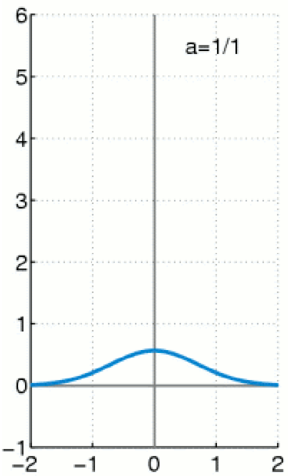
The Dirac Delta Function

The Dirac delta function, $\delta(\omega)$, is defined as:

- $\delta(\omega) = 0$ for all ω other than $\omega = 0$.
- $\delta(0) = \infty$
- The integral of $\delta(\omega)$, from any negative ω to any positive ω, is exactly 1:

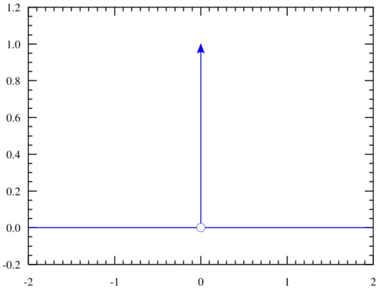
$$\int_{-\epsilon}^{\epsilon} \delta(\omega) d\omega = 1$$

It's useful to imagine the Dirac delta function as a tall, thin function — a Gaussian, a rectangle, or whatever — with zero width, infinite height, and an area of exactly 1.



https://commons.wikimedia.org/wiki/File:Dirac_function_approximation.gif

We usually draw it like this. The arrow has zero width, infinite height, and an area of exactly 1.0.



Integrating a Dirac Delta

The key use of a Dirac delta is that, when we multiply it by any function and integrate,

- All the values of that function at $\omega \neq 0$ are multiplied by $\delta(\omega) = 0$
- The value at $\omega = 0$ is multiplied by $+\infty$, in such a way that the integral is exactly:

$$\int_{-\pi}^{\pi} f(\omega)\delta(\omega)d\omega = f(0)$$

Integrating a Shifted Dirac Delta

The delta function can also be shifted, to some frequency ω_0. This is written as $\delta(\omega-\omega_0)$.

- All the values of that function at $\omega\neq\omega_0$ are multiplied by $\delta(\omega-\omega_0)=0$
- The value at $\omega=\omega_0$ is multiplied by $+\infty$, in such a way that the integral is exactly:

$$\int_{-\pi}^{\pi} f(\omega)\delta(\omega-\omega_0)d\omega = f(\omega_0)$$

Inverse DTFT of a Shifted Dirac Delta

Thus, for example,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(\omega-\omega_0)e^{j\omega n}d\omega = \frac{1}{2\pi}e^{j\omega_0 n}$$

In other words, the inverse DTFT of $Y(\omega)=\delta(\omega-\omega_0)$ is $y[n]=\frac{1}{2\pi}e^{j\omega_0 n}$, a complex exponential.

DTFT Pairs

By the linearity of the DTFT, we therefore have the following useful DTFT pairs:

$$e^{j\omega_0 n} \quad \leftrightarrow \quad 2\pi\delta(\omega - \omega_0),$$

and

$$\cos(\omega_0 n) \quad \leftrightarrow \quad \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Why This Answer Makes Sense

Suppose we were to try to find the DTFT of $x[n] = e^{j\omega_0 n}$ directly:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega-\omega_0)n}$$

- At frequencies $\omega \neq \omega_0$, we would be adding the samples of a sinusoid, which would give us $X(\omega) = 0$.
- At $\omega = \omega_0$, the summation becomes

$$X(\omega_0) = \sum_{n=-\infty}^{\infty} 1 = \infty$$

- So $X(\omega_0) = \infty$, and $X(\omega) = 0$ everywhere else. So it's a Dirac delta! The only thing the forward transform **doesn't** tell us is: **what kind of infinity?**

Why This Answer Makes Sense

- So $X(\omega_0)=\infty$, and $X(\omega)=0$ everywhere else. So it's a Dirac delta! The only thing the forward transform **doesn't** tell us is: **what kind of infinity?**
- The inverse DTFT gives us the answer. It needs to be the kind of infinity such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega = e^{j\omega_0 n},$$

and the solution is $X(\omega)=2\pi\delta(\omega-\omega_0)$

Outline

Windowing in Time = Convolution in Frequency

Remember that windowing in time = convolution in frequency:

$$y[n] = x[n]w[n] \quad \leftrightarrow \quad Y(\omega) = \frac{1}{2\pi}X(\omega) * W(\omega).$$

But if $x[n] = \cos(\omega_0 n)$, we already know that

$$y[n] = \cos(\omega_0 n)w[n] \quad \leftrightarrow \quad Y(\omega) = \frac{1}{2}W(\omega - \omega_0) + \frac{1}{2}W(\omega + \omega_0)$$

Can we reconcile these two facts?

Convolving with a Dirac delta function

The delta function is defined by this sampling property:

$$\int_{-\pi}^{\pi} \delta(\omega-\omega_0) f(\omega) d\omega = f(\omega_0)$$

What does that mean about convolution? Let's try it:

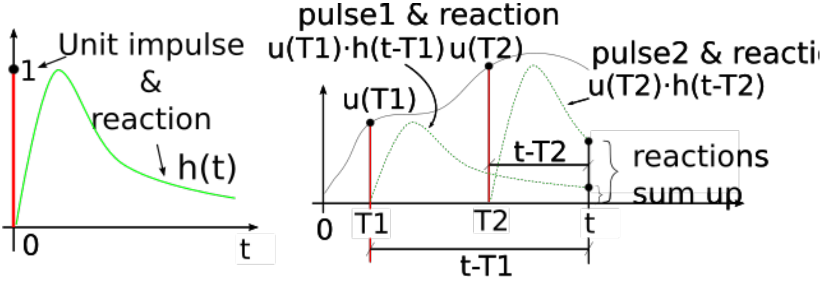
$$\delta(\omega-\omega_0) * W(\omega) = \int_{-\pi}^{\pi} \delta(\theta-\omega_0) W(\omega-\theta) d\theta$$
$$= W(\omega-\omega_0$$

Convolving with a Dirac delta function

So we see that:

$$\delta(\omega-\omega_0) * W(\omega) = W(\omega-\omega_0)$$

This is just like the behavior of impulses in the time domain:



Public domain,

https://commons.wikimedia.org/wiki/File:Convolution_of_two_pulses_with_impulse_response.svg

DTFT of a Windowed Cosine

So if:

$$\cos(\omega_0 n) \quad \leftrightarrow \quad \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0),$$

and

$$y[n]=x[n]w[n] \quad \leftrightarrow \quad Y(\omega)=\frac{1}{2\pi}X(\omega)*W(\omega),$$

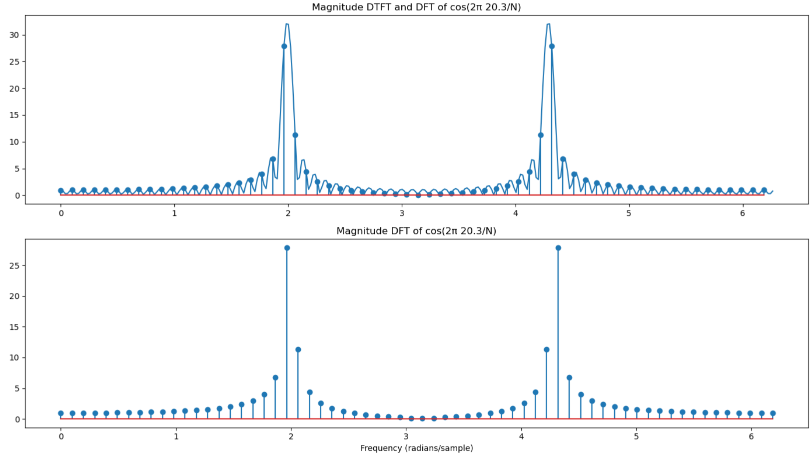
then

$$\begin{aligned}\cos(\omega_0 n)w[n] \quad &\leftrightarrow \quad \left(\frac{1}{2}\delta(\omega-\omega_0)*W(\omega)+\frac{1}{2}\delta(\omega+\omega_0)*W(\omega)\right)\\ &=\left(\frac{1}{2}W(\omega-\omega_0)+\frac{1}{2}W(\omega+\omega_0)\right)\end{aligned}$$

DFT of a Cosine

So again, we discover that:

$$x[n] = \cos\left(\frac{2\pi 20.3}{N}n\right)w[n]$$



Try the quiz!

Go to the course web page, and try the quiz!

Outline

Summary

- DTFT of a complex exponential is a delta function:

$$e^{j\omega_0 n} \quad \leftrightarrow \quad 2\pi\delta(\omega-\omega_0)$$

- DTFT of a cosine is two delta functions:

$$\cos(\omega_0 n) \quad \leftrightarrow \quad \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

- DTFT of a windowed cosine is frequency-shifted window functions:

$$\cos(\omega_0 n)w[n] \quad \leftrightarrow \quad \frac{1}{2}W(\omega-\omega_0)+\frac{1}{2}W(\omega+\omega_0)$$

Outline

Written Example

Consider the function

$$x[n] = A\cos(\omega_0 n + \theta)$$

What is $X(\omega)$?
How about $y[n] = w[n]x[n]$. What is $Y(\omega)$?