

# Lecture 24: Circular Convolution

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ECE 401: Signal and Image Analysis

- 1 Review: DTFT and DFT
- 2 Sampled in Frequency  $\leftrightarrow$  Periodic in Time
- 3 Circular Convolution
- 4 Zero-Padding
- 5 Summary

# Outline

- 1 Review: DTFT and DFT
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# Review: DTFT

The DTFT (discrete time Fourier transform) of any signal is  $X(\omega)$ , given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n - n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

# Properties of the DTFT

Properties worth knowing include:

- 0 Periodicity:  $X(\omega + 2\pi) = X(\omega)$
- 1 Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

- 2 Time Shift:  $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
- 3 Frequency Shift:  $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
- 4 Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

# Review: DFT

The DFT (discrete Fourier transform) of any signal is  $X[k]$ , given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi kn}{N}}$$
$$x[n] = \frac{1}{N} \sum_0^{N-1} X[k] e^{j \frac{2\pi kn}{N}}$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F[k] = 1$$
$$g[n] = \delta[((n - n_0))_N] \leftrightarrow G[k] = e^{-j \frac{2\pi kn_0}{N}}$$

# Properties of the DTFT

Properties worth knowing include:

- ① Periodicity:  $X[k + N] = X[k]$
- ① Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z[k] = aX[k] + bY[k]$$

- ② Circular Time Shift:  $x[((n - n_0))_N] \leftrightarrow e^{-j\frac{2\pi kn_0}{N}} X(\omega)$
- ③ Frequency Shift:  $e^{j\frac{2\pi k_0 n}{N}} x[n] \leftrightarrow X[k - k_0]$
- ④ Filtering is Circular Convolution:

$$y[n] = h[n] \circledast x[n] \leftrightarrow Y[k] = H[k]X[k],$$

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# Two different ways to think about the DFT

1.  $x[n]$  is finite length; DFT is samples of DTFT

$$x[n] = 0, n < 0 \text{ or } n \geq N \quad \leftrightarrow \quad X[k] = X(\omega)|_{\omega = \frac{2\pi k}{N}}$$

2.  $x[n]$  is periodic; DFT is scaled version of Fourier series

$$x[n] = x[n + N] \quad \leftrightarrow \quad X[k] = NX_k$$

# 1. $x[n]$ finite length, DFT is samples of DTFT

If  $x[n]$  is nonzero only for  $0 \leq n \leq N - 1$ , then

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n},$$

and

$$X[k] = X(\omega)|_{\omega=\frac{2\pi k}{N}}$$

## 2. $x[n]$ periodic, $X[k] = NX_k$

If  $x[n] = x[n + N]$ , then its Fourier series is

$$X_k = \frac{1}{N} \sum_{n=1}^{N-1} x[n] e^{-j \frac{2\pi kn}{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j \frac{2\pi kn}{N}},$$

and its DFT is

$$X[k] = \sum_{n=1}^{N-1} x[n] e^{-j \frac{2\pi kn}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi kn}{N}}$$

# Delayed impulse wraps around

$$\delta [((n - n_0))_N] \leftrightarrow e^{-j\frac{2\pi kn_0}{N}}$$

# Delayed impulse is really periodic impulse

$$\delta [((n - n_0))_N] \leftrightarrow e^{-j\frac{2\pi kn_0}{N}}$$

# Principal Phase

- Something weird going on: how can the phase keep getting bigger and bigger, but the signal wraps around?
- It's because the phase wraps around too!

$$\angle X[k] = -\omega_k(N + n) = -\omega_k n, \quad \text{because } \omega_k = \frac{2\pi k}{N}$$

- **Principal phase** = add  $\pm 2\pi$  to the phase, as necessary, so that  $-\pi < \angle X[k] \leq \pi$
- **Unwrapped phase** = let the phase be as large as necessary so that it is plotted as a smooth function of  $\omega$

# Unwrapped phase vs. Principal phase

$$\delta [((n - n_0))_N] \leftrightarrow e^{-j\frac{2\pi kn_0}{N}}$$

# Summary: Two different ways to think about the DFT

1.  $x[n]$  is finite length; DFT is samples of DTFT

$$x[n] = 0, n < 0 \text{ or } n \geq N \quad \leftrightarrow \quad X[k] = X(\omega)|_{\omega = \frac{2\pi k}{N}}$$

2.  $x[n]$  is periodic; DFT is scaled version of Fourier series

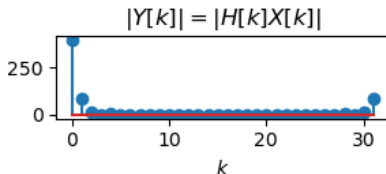
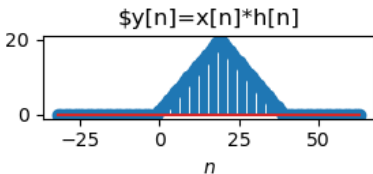
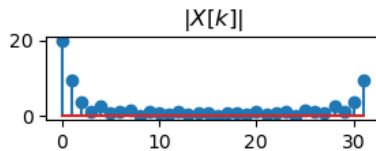
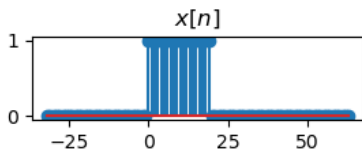
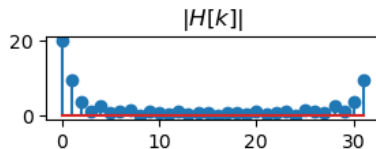
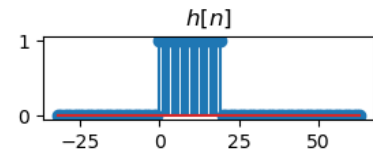
$$x[n] = x[n + N] \quad \leftrightarrow \quad X[k] = NX_k$$



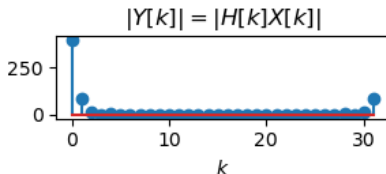
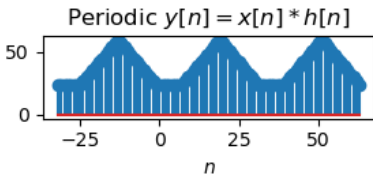
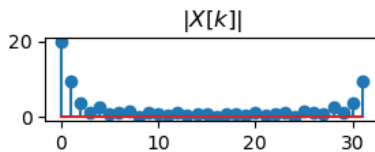
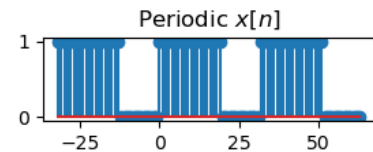
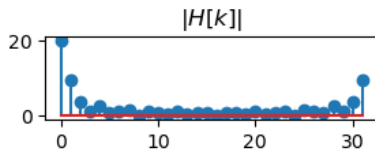
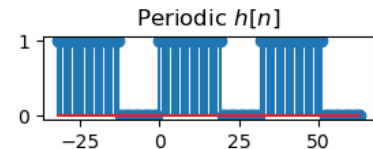
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# Multiplying two DFTs: what we think we're doing



# Multiplying two DFTs: what we're actually doing



# Circular convolution

Suppose  $Y[k] = H[k]X[k]$ , then

$$\begin{aligned}y[n] &= \frac{1}{N} \sum_{k=0}^{N-1} H[k]X[k]e^{j\frac{2\pi kn}{N}} \\&= \frac{1}{N} \sum_{k=0}^{N-1} H[k] \left( \sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi km}{N}} \right) e^{j\frac{2\pi kn}{N}} \\&= \sum_{m=0}^{N-1} x[m] \left( \frac{1}{N} \sum_{k=0}^{N-1} H[k]e^{-j\frac{2\pi k(n-m)}{N}} \right) \\&= \sum_{m=0}^{N-1} x[m]h[\left((n-m)\right)_N]\end{aligned}$$

The last line is because  $\frac{2\pi k(n-m)}{N} = \frac{2\pi k((n-m))_N}{N}$ .

# Circular convolution

Multiplying the DFT means **circular convolution** of the time-domain signals:

$$y[n] = h[n] \circledast x[n] \leftrightarrow Y[k] = H[k]X[k],$$

Circular convolution ( $h[n] \circledast x[n]$ ) is defined like this:

$$h[n] \circledast x[n] = \sum_{m=0}^{N-1} x[m]h[((n-m))_N] = \sum_{m=0}^{N-1} h[m]x[((n-m))_N]$$

# Circular convolution example

# Circular convolution example

# Quiz!

Go to the course webpage, try the quiz!

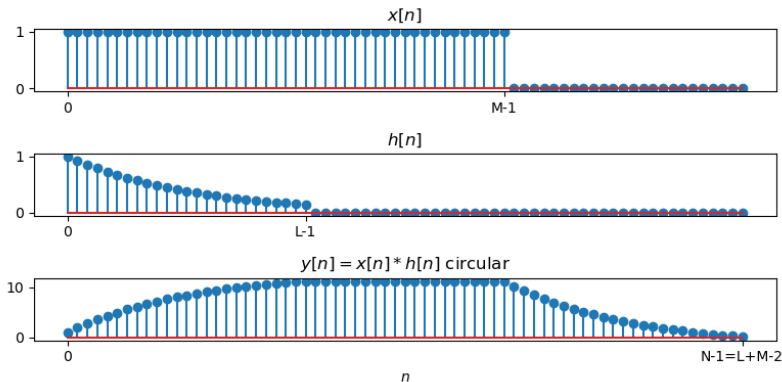


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# How long is $h[n] * x[n]$ ?

If  $x[n]$  is  $M$  samples long, and  $h[n]$  is  $L$  samples long, then their linear convolution,  $y[n] = x[n] * h[n]$ , is  $M + L - 1$  samples long.



# Zero-padding turns circular convolution into linear convolution

How it works:

- $h[n]$  is length- $L$
- $x[n]$  is length- $M$
- As long as they are both zero-padded to length  $N \geq L + M - 1$ , then
- $y[n] = h[n] \circledast x[n]$  is the same as  $h[n] * x[n]$ .

# Zero-padding turns circular convolution into linear convolution

Why it works: Either...

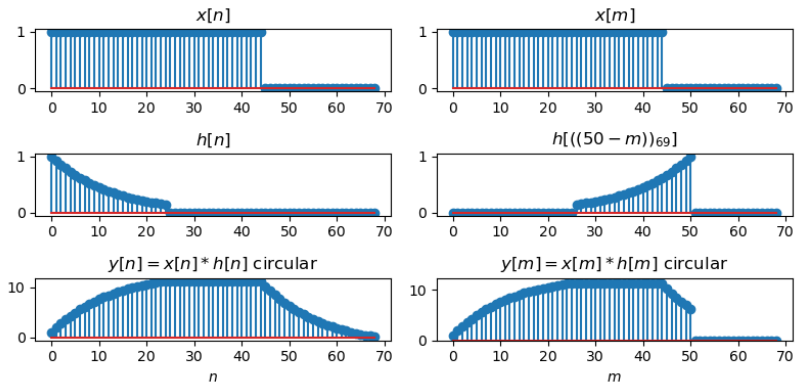
- $n - m$  is a positive number, between 0 and  $N - 1$ . Then  $((n - m))_N = n - m$ , and therefore

$$x[m]h[((n - m))_N] = x[m]h[n - m]$$

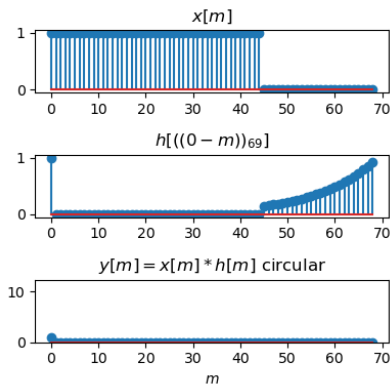
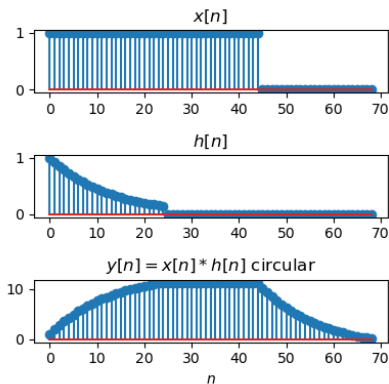
- $n - m$  is a negative number, between 0 and  $-(L - 1)$ . Then  $((n - m))_N = N + n - m \geq N - (L - 1) > M - 1$ , so

$$x[m]h[((n - m))_N] = 0$$

Case #1:  $n - m$  is positive, so circular convolution is the same as linear convolution



Case #2:  $n - m$  is negative, so it wraps around, but  $N$  is long enough so that the wrapped part of  $h[((n - m))_N]$  doesn't overlap with  $x[m]$



# Zero-padding turns circular convolution into linear convolution

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Circular convolution is the same as linear convolution if and only if  $N \geq L + M - 1$ .