Lecture 17: Averaging Filter, a.k.a. Scaled Rectangular Window

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ECE 401: Signal and Image Analysis

Outline

Review: Convolution

- A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It's defined as:

$$y[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

- We use the symbol $*$ to mean "convolution:"

$$y[n] = g[n] * f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

Frequency Response

- **Tones in → Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = G(\omega)e^{j\omega n}$$

$$x[n] = \cos(\omega n) \rightarrow y[n] = |G(\omega)| \cos(\omega n + \angle G(\omega))$$

$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|G(\omega)| \cos(\omega n + \theta + \angle G(\omega))$$

- where the **Frequency Response** is given by

$$G(\omega) = \sum_m g[m]e^{-j\omega m}$$

Review: Fourier Series

In continuous time, any periodic signal can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t},$$

where

$$X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi k F_0 t} dt$$

Review: DT Processing of CT Signals

A bandlimited periodic signal $x(t)$ can be sampled, filtered, then sinc-interpolated to create:

$$x(t) = \sum_{k=-N}^{N} X_k e^{j2\pi k F_0 t}$$

$$x[n] = \sum_{k=-N}^{N} X_k e^{jk\omega_0 n},$$

$$y[n] = \sum_{k=-N}^{N} Y_k e^{jk\omega_0 n},$$

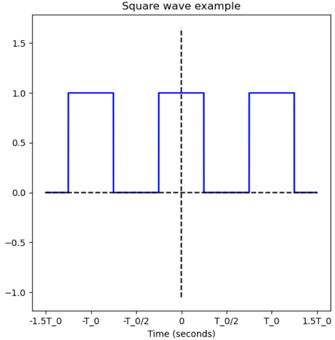
$$y(t) = \sum_{k=-N}^{N} Y_k e^{j2\pi k F_0 n},$$

where $\omega_0 = \frac{2\pi F_0}{F_s}$, and $N = \lfloor \frac{F_s/2}{F_0} \rfloor$, and $Y_k = H(k\omega_0)X_k$.

Outline

Square wave example

Let's use a square wave with a nonzero DC value, like this one:



$$x(t) = \begin{cases} 1 & -\frac{L}{2} < t < \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

. . . where L is the length of the nonzero part, in seconds.

Fourier Series

Analysis (finding the spectrum, given the waveform):

$$X_k = \begin{cases} \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)dt & k=0 \\ \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi kt/T_0}dt & k\neq 0 \end{cases}$$

$$= \begin{cases} \frac{1}{T_0}\int_{-L/2}^{L/2} dt & k=0 \\ \frac{1}{T_0}\int_{-L/2}^{L/2} e^{-j2\pi kt/T_0}dt & k\neq 0 \end{cases}$$

$$= \begin{cases} \frac{L}{T_0} & k=0 \\ \frac{\sin(\pi kL/T_0)}{\pi k} & k\neq 0 \end{cases}$$

Outline

Discrete-Time Fourier Series

A signal that's periodic in discrete time also has a Fourier series. If the signal is periodic with a period of $N_0 = T_0 F_s$ samples, then its Fourier series is

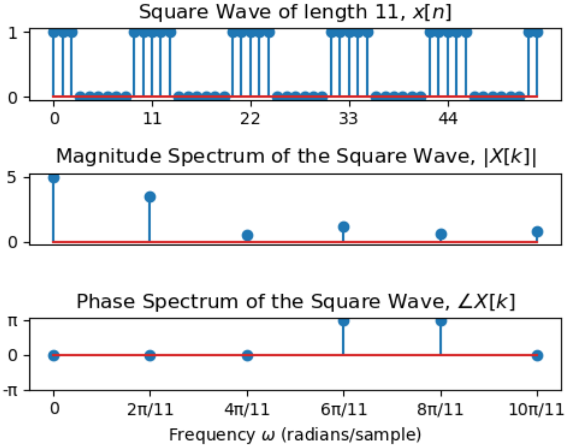
$$x[n] = \sum_{k=0}^{N_0-1} X_k e^{j2\pi kn/N_0} = \sum_{k=-N_0/2}^{(N_0-1)/2} X_k e^{j2\pi kn/N_0}$$

and the Fourier analysis formula is

$$X_k = \frac{1}{N_0} \sum_{n=-N_0/2}^{N_0/2-1} x[n] e^{-j2\pi kn/N_0}$$

Example: Spectrum of a Square Wave

For example, here's an even-symmetric ($x[n] = x[-n]$) square wave with a period of $N_0 = 11$ samples and a length of $L = 5$ samples, i.e., $x[n] = 1$ for $-\frac{L-1}{2} \leq n \leq \frac{L-1}{2}$:



Spectrum of a Square Wave

The Fourier series coefficients of this square wave are

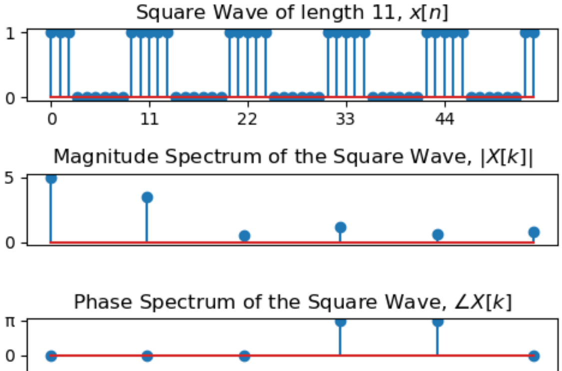
$$X_k = \begin{cases} \frac{1}{N_0}\sum_{-N_0/2}^{N_0/2-1} x[n] & k=0 \\ \frac{1}{N_0}\sum_{-N_0/2}^{N_0/2-1} x[n]e^{-j2\pi kn/N_0} & k\neq 0 \end{cases}$$

$$= \begin{cases} \frac{1}{N_0}\sum_{-(L-1)/2}^{(L-1)/2} 1 & k=0 \\ \frac{1}{N_0}\sum_{-(L-1)/2}^{(L-1)/2} e^{-j2\pi kn/N_0} & k\neq 0 \end{cases}$$

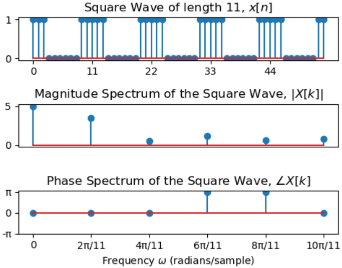
Spectrum of a Square Wave

The Fourier series coefficients of this square wave are approximately, but not exactly, the same as they would be in the continuous-time case:

$$X_k \approx \begin{cases} \frac{L}{N_0} & k = 0 \\ \frac{\sin(\pi k L/N_0)}{\pi k} & k \neq 0 \end{cases}$$



More about the phase spectrum

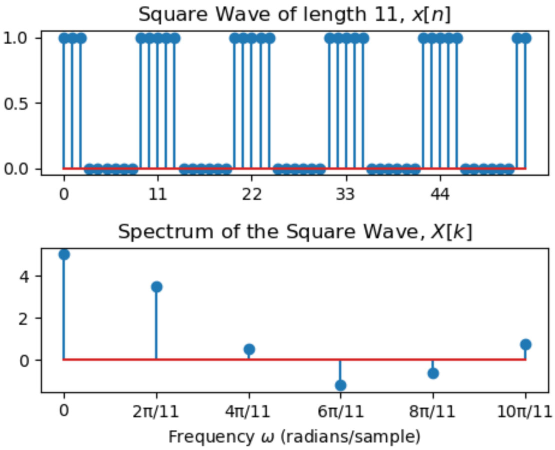


Notice that, for the phase spectrum of a square wave, the phase spectrum is either $\angle X[k] = 0$ or $\angle X[k] = \pi$. That means that the spectrum is real-valued, with no complex part:

- **Positive real:** $X[k] = |X[k]|$
- **Negative real:** $X[k] = -|X[k]| = |X[k]|e^{j\pi}$

More about the phase spectrum

Having discovered that the square wave has a real-valued $X[k]$, we could just plot $X[k]$ itself, instead of plotting its magnitude and phase:



Fourier Series and Fourier Transform

Notice that, for both the continuous-time and discrete-time Fourier series, the square wave has three basic properties:

- X_0 is just the average value of $x(t)$.
- X_k is (exactly or approximately) $\sin(\pi kL/T_0)/\pi k$.
- If the square wave is symmetric in time, then X_k is real-valued.

Now let's see how those properties generalize to the DTFT of a non-periodic rectangle.

Outline

Local Average Filters

Let's go back to the local averaging filter. I want to define two different types of local average: centered, and delayed.

- **Centered local average:** This one averages $\left(\frac{L-1}{2}\right)$ future samples, $\left(\frac{L-1}{2}\right)$ past samples, and $x[n]$:

$$y_c[n] = \frac{1}{L} \sum_{m=-\left(\frac{L-1}{2}\right)}^{\left(\frac{L-1}{2}\right)} x[n-m]$$

- **Causal local average:** This one averages $x[n]$ and $L-1$ of its past samples:

$$y_d[n] = \frac{1}{L} \sum_{m=0}^{L-1} x[n-m]$$

Notice that $y_d[n] = y_c\left[n - \left(\frac{L-1}{2}\right)\right]$.

Local Average Filters

We can write both of these as filters:

- **Centered local average:**

$$y_c[n] = f_c[n] * x[n]$$

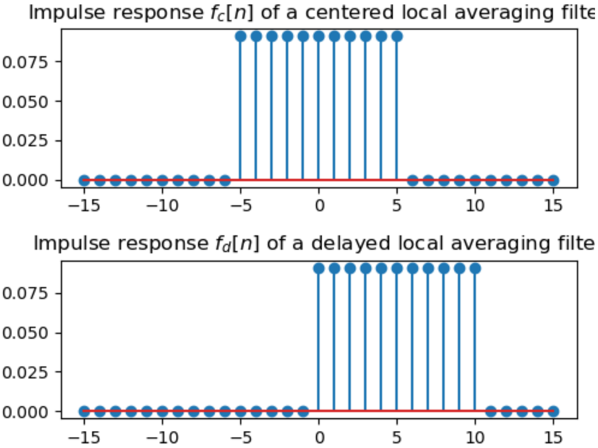
$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

- **Causal local average:**

$$y_d[n] = f_d[n] * x[n]$$

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Local Average Filters



Notice that $f_d[n] = f_c\left[n - \left(\frac{L-1}{2}\right)\right]$.

The relationship between centered local average and delayed local average

Notice that $f_d[n] = f_c[n - \frac{L-1}{2}]$. We can find the relationship between their DTFTs using variable substitution, with the variable $m = n - \frac{L-1}{2}$:

$$\begin{aligned} F_d(\omega) &= \sum_{n=-\infty}^{\infty} f_d[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} f_c[n - \frac{L-1}{2}] e^{-j\omega n} \\ &= e^{-j\omega \frac{L-1}{2}} \sum_{m=-\infty}^{\infty} f_c[m] e^{-j\omega m} \\ &= e^{-j\omega \frac{L-1}{2}} F_c(\omega) \end{aligned}$$

The frequency response of a local average filter

Let's find the frequency response of

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

The formula is

$$F_d(\omega) = \sum_m f[m] e^{-j\omega m},$$

so,

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

The frequency response of a local average filter

$$F_d(\omega) = \sum_{m=0}^{L-1} \frac{1}{L} e^{-j\omega m}$$

This is just a standard geometric series,

$$\sum_{m=0}^{L-1} a^m = \frac{1-a^L}{1-a},$$

so:

$$F_d(\omega) = \frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right)$$

The frequency response of a local average filter

We now have an extremely useful transform pair:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right)$$

Let's attempt to convert that into polar form, so we can find magnitude and phase response. Notice that both the numerator and the denominator are subtractions of complex numbers, so we might be able to use $2j\sin(x) = e^{jx} - e^{-jx}$ for some x. Let's try:

$$\begin{aligned} \frac{1}{L}\left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right) &= \frac{1}{L}\frac{e^{-j\omega L/2}}{e^{-j\omega/2}}\left(\frac{e^{j\omega L/2}-e^{-j\omega L/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) \\ &= e^{-j\omega\left(\frac{L-1}{2}\right)}\frac{1}{L}\left(\frac{2j\sin(\omega L/2)}{2j\sin(\omega/2)}\right) \\ &= e^{-j\omega\left(\frac{L-1}{2}\right)}\frac{1}{L}\left(\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right) \end{aligned}$$

Quiz

Go to the course web page, and try the quiz!

The frequency response of a local average filter

Now we have $F_d(\omega)$ in almost magnitude-phase form:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le m \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \left(\frac{\sin(\omega L/2)}{L\sin(\omega/2)}\right) e^{-j\omega\left(\frac{L-1}{2}\right)}$$

By the way, remember we discovered that

$$F_d(\omega) = e^{-j\omega\left(\frac{L-1}{2}\right)} F_c(\omega)$$

Notice anything?

DTFT of Local Averaging Filters

- **Centered local average:**

$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

$$F_c(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)}$$

- **Causal local average:**

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad F_d(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)} e^{-j\omega\left(\frac{L-1}{2}\right)}$$

Outline

DTFT of Rectangular Window

The local summing filter is just a scaled version of the local averaging filter. It is commonly called the "rectangular window," for reasons that we'll explore in future lectures.

- **Centered rectangular window:**

$$w_c[n] = \begin{cases} 1 & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

$$W_c(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

- **Causal rectangular window:**

$$w_d[n] = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad W_d(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega\left(\frac{L-1}{2}\right)}$$

Dirichlet form

Of all four of these signals, the centered rectangular window has the simplest DTFT. The textbook calls this signal the "Dirichlet form:"

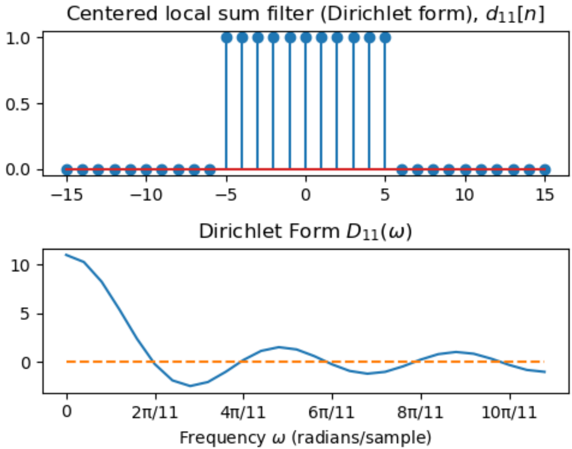
$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

That is, exactly, the frequency response of a centered rectangular window:

$$d_L[n] = w_c[n] = \begin{cases} 1 & -\left(\frac{L-1}{2}\right) \leq n \leq \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

Dirichlet form

Here's what it looks like:



Dirichlet form

Since every local averaging filter is based on Dirichlet form, it's worth spending some time to understand it better.

$$D_L(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

- It's equal to zero every time $\omega L/2$ is a multiple of π. So

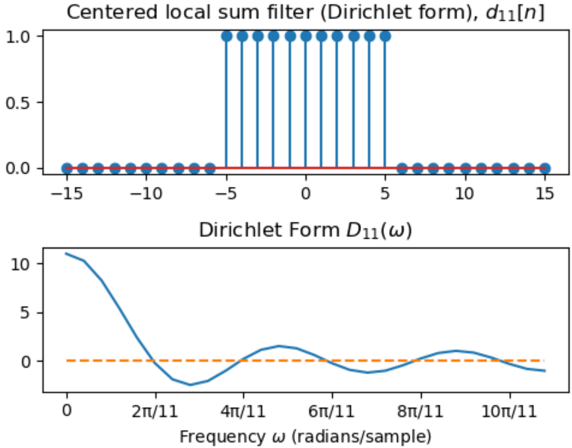
$$D_L\left(\frac{2\pi k}{L}\right) = 0 \quad \text{for all integers } k \text{ except } k = 0$$

- At $\omega = 0$, the value of $\frac{\sin(\omega L/2)}{\sin(\omega/2)}$ is undefined, but it's posssible to prove that $\lim_{\omega\to 0} D_L(\omega) = L$. To make life easy, we'll just define it that way:

$$\text{DEFINE:} \quad D_L(0) = L$$

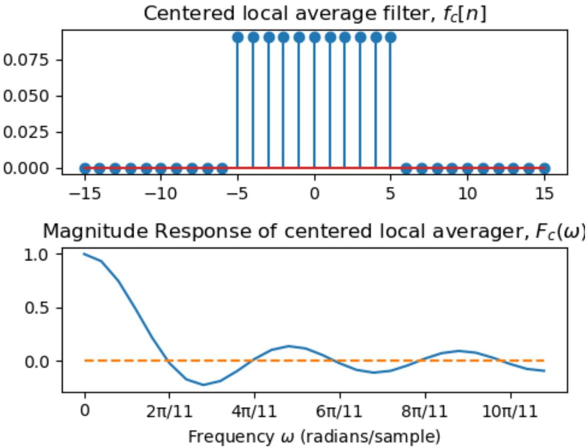
Dirichlet form

Here's what it looks like:



Local averaging filter

Here's what the centered local averaging filter looks like. Notice that it's just $1/L$ times the Dirichlet form:



Outline

Summary: DTFTs of Rectangular Windows

- The **Centered Local Averaging Filter** is $1/L$ times the Dirichlet form:

$$f_c[n] = \begin{cases} \frac{1}{L} & -\left(\frac{L-1}{2}\right) \le n \le \left(\frac{L-1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_c(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)}$$

- The **Causal Local Averaging Filter** is $f_c[n]$, delayed by $\frac{L-1}{2}$ samples:

$$f_d[n] = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F_d(\omega) = \frac{\sin(\omega L/2)}{L\sin(\omega/2)} e^{-j\omega\left(\frac{L-1}{2}\right)}$$