Lecture 16: Causality and Stability

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ECE 401: Signal and Image Analysis

Overview: Causality and Stability

- A system is causal if and only if $h[n]$ is right-sided.
- A system is causal only if the unwrapped generalized phase is non-positive, $\phi(\omega) \leq 0$
- A system is stable if and only if $h[n]$ is magnitude-summable.
- A system is stable only if $|H(\omega)|$ is finite for all ω

1. Review: Impulse Response and Frequency Response
2. Causality = The future is unknown
3. Stability = All finite inputs produce finite outputs
4. Summary

Outline

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Impulse Response and Convolution

The impulse response of a system is its response to an impulse:

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$

If a system is linear and shift-invariant, then its output, in response to **any** input, can be computed using convolution:

$$x[n] \xrightarrow{\mathcal{H}} y[n] = h[n] * x[n]$$

Frequency Response

The frequency response of a system is its response to a pure tone:

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos(\omega n + \measuredangle H(\omega))$$

The frequency response is related to the impulse response by:

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Outline

Causality

Definition: A **causal** system is a system whose output at time n, $y[n]$, depends on inputs $x[m]$ only for $m \leq n$.

What systems are causal? What systems are non-causal?

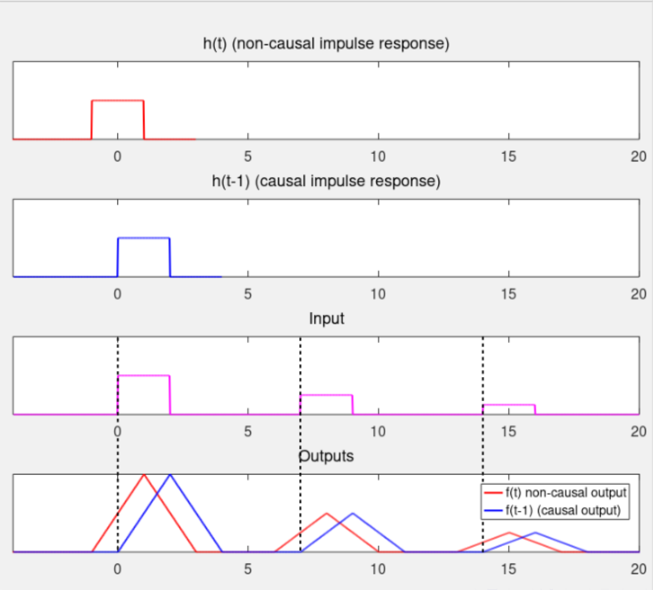
- A real-time system must be causal.
- If n is time, but the system is operating in batch mode, then it doesn't need to be causal.
- If n is space (e.g., rows or columns of an image), the system doesn't need to be causal.

Causal system $\Leftrightarrow$ Right-sided impulse response

$$y[n] = \sum_m h[m]x[n-m]$$

- This system is **causal** iff $y[n]$ depends on $x[n-m]$ only for $n - m \leq n$.
- In other words, the system is causal iff $h[m] = 0$ for all $m < 0$.

Causal system $\Leftrightarrow$ Right-sided impulse response



Variations on the word "causal"

- A **causal** system is one that depends only on the present and the past, i.e., $h[n]$ is right-sided.
- A **non-causal** system is one that's not causal.
- A **anti-causal** system is one that depends only on the present and the future, i.e., $h[n]$ is left-sided.

Causality $\Rightarrow$ Non-Positive Unwrapped Generalized Phase

Remember how we can calculate $H(\omega)$:

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = R(\omega)e^{j\phi(\omega)}$$

- If the system is causal, then the only nonzero terms in that sum are terms with non-positive phase ($e^{-j\omega m}$)
- Therefore causal systems have an unwrapped generalized phase $\phi(\omega) \leq 0$

Causality $\Rightarrow$ Non-Positive Unwrapped Generalized Phase

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = |H(\omega)|e^{j\measuredangle H(\omega)}$$

There are two ways in which a causal system can have $\measuredangle H(\omega) > 0$:

1. $\measuredangle H(\omega)$ is the principal angle, i.e., $-\pi < \measuredangle H(\omega) \leq \pi$; it wraps around the circle in order to satisfy that constraint
2. $|H(\omega)| \geq 0$, so the only way we can represent negative numbers is using $e^{j\pi} = -1$, i.e., by adding $\pm\pi$ to the phase.

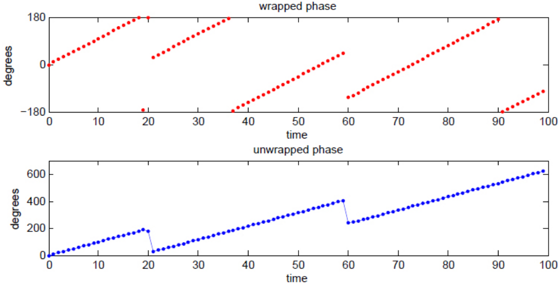
The "unwrapped generalized phase," $\phi(\omega)$, simplifies both of those edge cases.

Unwrapped Phase

The "unwrapped phase" of an LTI system can be defined as

$$\angle H(\omega) = \underset{k}{\operatorname{argmin}} \left| \frac{d}{d\omega} \left(\measuredangle H(\omega) + k2\pi \right) \right|$$

In other words, add or subtract 2π to each value of $\measuredangle H(\omega)$ in order to make it as nearly as possible a continuous function of ω. Here's an example (for an anti-causal system: note that the remaining discontinuities in $\angle H(\omega)$ are because the lower plot is **unwrapped** but not **generalized**):



Unwrapped Generalized Phase

The "unwrapped generalized phase response" of an LTI system, $\phi(\omega)$, is the phase response, ignoring the $\pm\pi$ shift that accounts for negative real parts. In other words,

$$\phi(\omega) = \underset{k}{\operatorname{argmin}} \left|\frac{d}{d\omega}\left(\measuredangle H(\omega) + k\pi\right)\right|$$

such that

$$H(\omega) = R(\omega)e^{j\phi(\omega)}$$

$$R(\omega) = \pm|H(\omega)| = \begin{cases} |H(\omega)| & R(\omega) \geq 0 \\ |H(\omega)|e^{j\pi} & R(\omega) < 0 \end{cases}$$

In other words, $\phi(\omega)$ is what $\angle H(\omega)$ would be if we allowed the magnitude response to be negative.

Causality $\Rightarrow$ Non-Positive Unwrapped Generalized Phase

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = R(\omega)e^{j\phi(\omega)}$$

If you put a cosine into a system, you get a cosine advanced by its phase response $\phi(\omega)$:

$$\cos(\omega n) \xrightarrow{\mathcal{H}} R(\omega)\cos\left(\omega n + \phi(\omega)\right)$$

- If $\phi(\omega) > 0$, it means that the output is happening **before** the input! So a system is causaul only if $\phi(\omega) \leq 0$.
- The converse is not true: non-causal systems can also have $\phi(\omega) \leq 0$.

Quiz

Go to the course webpage, and try the quiz!

Overview: Causality and Stability

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- A system is stable only if $|H(\omega)|$ is finite for all ω

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Stability

Definition: A system is **stable** if and only if **every** bounded $x[n]$ (every signal such that $|x[n]| < \infty$ for all n) produces a bounded output ($|y[n]| < \infty$ for all n).

Why Stability Matters

- If your system is unstable, then every now and then, you'll get an inexplicable bug:
- all of the samples of $y[n]$ will be `FLT_MAX`!
- That's very hard to debug. If you view it as an image, or listen to it, it will sound like you just didn't generate the samples, so you will be looking for the error in the wrong place!

Magnitude-summable impulse response $\Rightarrow$ Stable system

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Suppose we know that $|x[n]| \leq M$, for some finite M, for all n. Then

$$|y[n]| \leq M \sum_{m=-\infty}^{\infty} |h[m]|$$

So

$$\sum_{m=-\infty}^{\infty} |h[m]| \text{ is finite } \Rightarrow \text{ System is stable}$$

Stable system $\Rightarrow$ Magnitude-summable impulse response

On the other hand, suppose that

$$\sum_{m=-\infty}^{\infty} |h[m]| = \infty$$

Does that mean that the system is **unstable**? Yes! Yes, it does! Consider the "worst-case" input

$$x[n] = \text{sign}\left(h[-n]\right)$$

Then $y[0]$ is

$$y[0] = \sum_{m=-\infty}^{\infty} h[m]x[-m] = \sum_{m=-\infty}^{\infty} |h[m]| = \infty$$

Example: Weighted Average

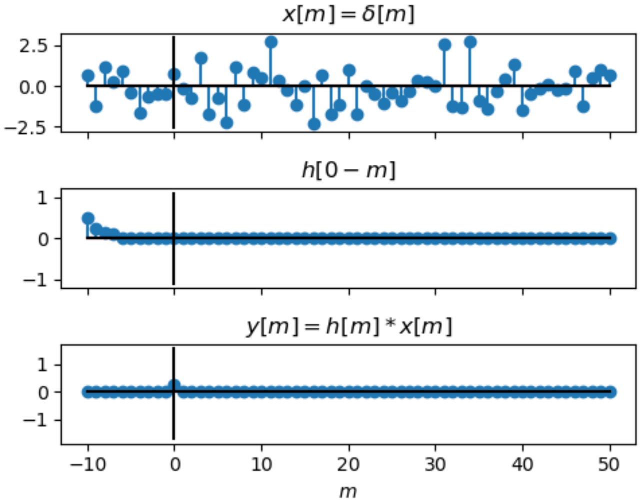
For example, consider a 7-tap weighted average:

$$y[n] = \sum_{m=-3}^{3} h[m]x[n-m]$$

As long as all of the weights are finite ($|h[m]| < \infty$ for all m), then $\sum_{m=-3}^{3} |h[m]|$ is also finite, so the system is stable

Example: Weighted Average

For any finite input, the output is finite:



Example: Summation

For example, consider summation:

$$y[n] = \sum_{m=0}^{\infty} x[n-m]$$

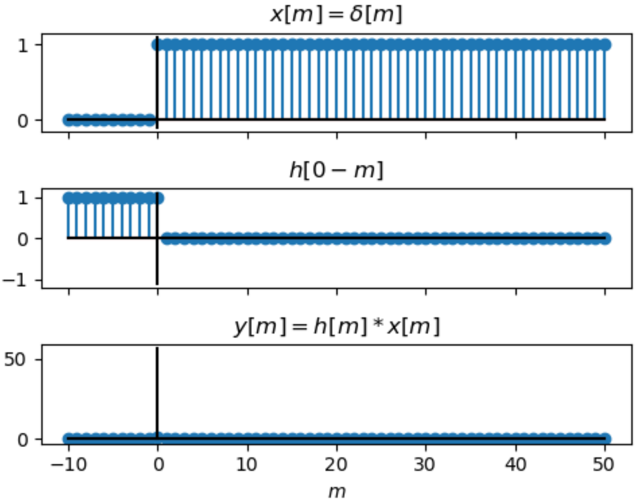
This is an **unstable** system!!

$$h[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 1 = \infty$$

Example: Summation

For example, if the input is a unit step, the output is unbounded:



Example: Obviously Unstable System

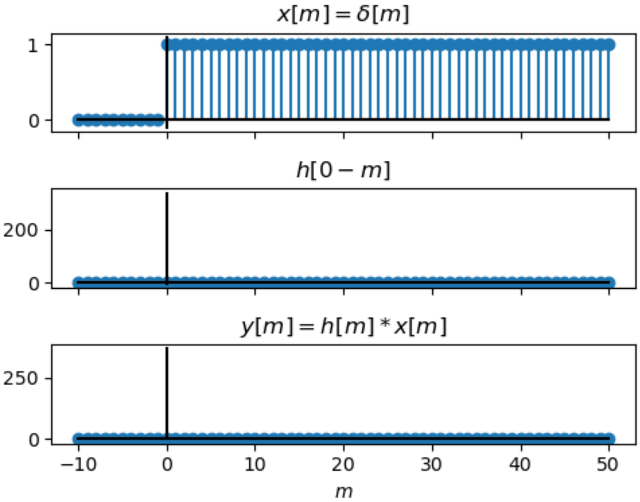
Finally, some systems are just obviously unstable. Consider

$$h[n] = (1.1)^n u[n]$$

This is obviously unstable. In fact, not only does $|h[n]|$ sum to infinity — it even goes to infinity if the input is just a delta function!

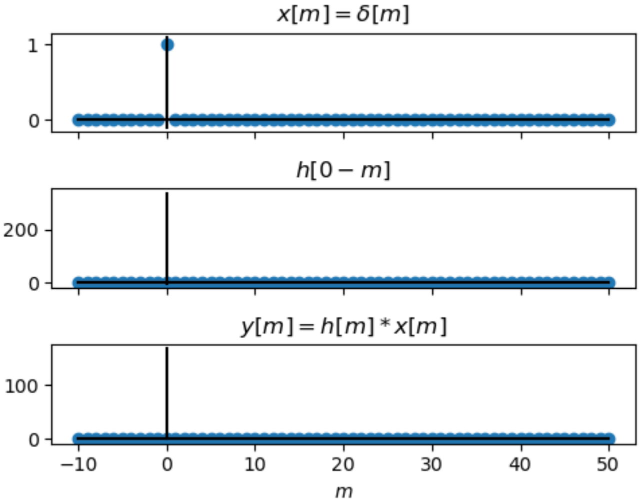
Example: Obviously Unstable System

For example, if the input is a unit step, the output is unbounded:



Example: Obviously Unstable System

Even if the input is a delta function, the output is unbounded:



Relationship to Frequency Response

How about the frequency response of stable versus unstable systems?

It turns out that a system is stable only if $|H(\omega)|$ is finite for all ω.

Proof:

$$\begin{aligned} |H(\omega)| &= \left|\sum_{m=-\infty}^{\infty} h[n]e^{-j\omega n}\right| \\ &\le \sum_{m=-\infty}^{\infty} |h[n]| \end{aligned}$$

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