# Lecture 14: DT Filtering of CT Signals

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ECE 401: Signal and Image Analysis

# Outline

# Sampling: Continuous Time $\rightarrow$ Discrete Time

A signal is sampled by measuring its value once every $T_s$ seconds:

$$x[n] = x(t = nT_s)$$

The spectrum of the DT signal has components at $\omega = \frac{2\pi f}{F_s}$, and also at every $2\pi\ell + \omega$ and $2\pi\ell - \omega$, for every integer $\ell$. Aliasing occurs unless $|\omega| \leq \pi$.

# Interpolation: Discrete Time → Continuous Time

A CT signal $y(t)$ can be created from a DT signal $y[n]$ by interpolation:

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

- $p(t) =$rectangle $\Rightarrow$ PWC interpolation
- $p(t) =$triangle $\Rightarrow$ PWL interpolation
- $p(t) = \text{sinc}\left(\frac{\pi t}{T_s}\right) \Rightarrow$ perfectly bandlimited interpolation, $y(t)$ has no spectral components above $F_N$

# Convolution

Convolution (finite impulse response filtering) is a generalization of weighted local averaging:

$$y[n] = h[n] * x[n] \equiv \sum_m x[m]h[n-m] = \sum_m x[n-m]h[m]$$

- If all samples of $h[n]$ are positive, then it's a weighted local averaging filter
- If the samples of $h[n]$ are positive for $n > 0$ and negative for $n < 0$ (or vice versa), then it's a weighted local differencing filter

# Frequency Response

- **Tones in → Tones out**

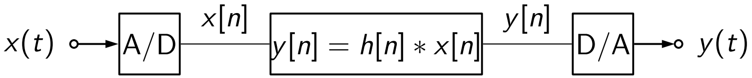
$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)|\cos(\omega n + \angle H(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)|\cos(\omega n + \theta + \angle H(\omega))$$

- where the **Frequency Response** is given by

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

# Outline

# DT Filtering of CT Signals




Constraints:

- Assume that A/D and D/A use same $F_s$
- Assume $F_s \geq 2f_{\max}$
- Assume sinc interpolation

# DT Filtering of CT Signals

- If $h[n]$ is a local averager, what's the relationship of $y(t)$ to $x(t)$?
- If $h[n]$ is a local differencer, what's the relationship of $y(t)$ to $x(t)$?

# Outline

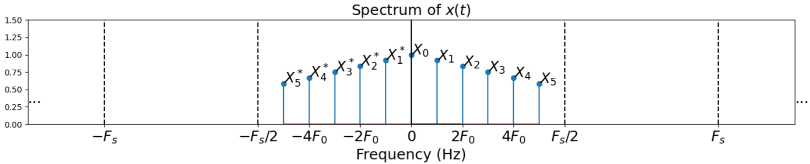
# Fourier Series

To start with, let's assume $x(t)$ is periodic and bandlimited, so:

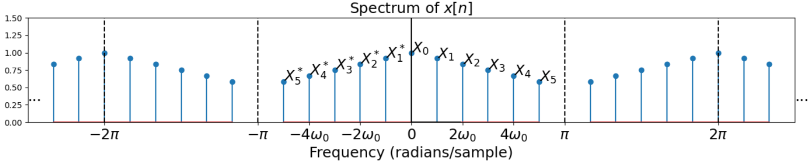
$$x(t) = \sum_{k=-N}^{N} X_k e^{j2\pi kF_0 t}$$
$$x[n] = \sum_{k=-N}^{N} X_k e^{jk\omega_0 n},$$

where $\omega_0 = \frac{2\pi F_0}{F_s}$, and $N = \lfloor \frac{F_s/2}{F_0} \rfloor$ is number of harmonics between 0 and the Nyquist frequency.

# Spectrum of $x(t)$



# Spectrum of $x[n]$



# Sampling of Periodic Bandlimited Signals

Now we know how to get from $x(t)$ to $x[n]$, assuming $x(t)$ is periodic and bandlimited:

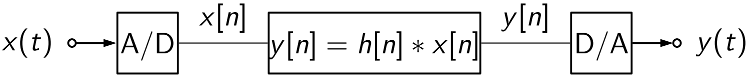
$$x(t) = \sum_{k=-N}^{N} X_k e^{j2\pi k F_0 t}$$

$$x[n] = \sum_{k=-N}^{N} X_k e^{jk\omega_0 n}$$

Next: What is $y[n]$?

# Outline

# DT Filtering of a Pure Tone




Let's focus on just one of the harmonics of $x(t)$:

$$x(t) = Xe^{j2\pi ft}$$
$$x[n] = Xe^{j\omega n}$$
$$y[n] = Ye^{j\omega n}$$
$$y(t) = Ye^{j2\pi ft}$$

Can we find the relationship between the two phasors $Y$ and $X$?

# Relationship Between $Y$ and $X$

Remember that $y[n] = x[n] * h[n]$, i.e.,

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

What happens if we plug in $x[n] = Xe^{j\omega n}$?

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]Xe^{j\omega(n-m)} \\ &= Xe^{j\omega n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \\ &= Xe^{j\omega n} H(\omega) \end{aligned}$$

So $Y = H(\omega)X$, where $H(\omega)$ is the frequency response!

# Filtering Periodic Signals

Now let's generalize to the case when $x(t)$ is any periodic bandlimited signal

$$x(t) = \sum_{k=-N}^{N} X_k e^{j2\pi k F_0 t}$$

$$x[n] = \sum_{k=-N}^{N} X_k e^{jk\omega_0 n}$$

Since the system is linear, adding inputs $\Rightarrow$ add outputs. So the output is

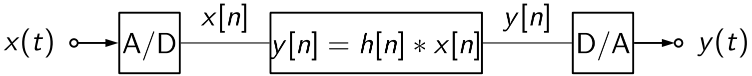
$$y[n] = \sum_{k=-N}^{N} Y_k e^{jk\omega_0 n}, \qquad = \sum_{k=-N}^{N} H(k\omega_0) X_k e^{jk\omega_0 n},$$

...where $H(k\omega_0)$ is the frequency response of the system at frequency $k\omega_0$.

# Outline

# DT Filtering of CT Signals

Remember, here's the overall system:




. . . and remember that the D/A uses ideal bandlimited sinc interpolation.

# Sinc Interpolation

Remember that the sinc is the function that perfectly reconstructs any signal below the Nyquist frequency:

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]\text{sinc}\left(\pi(t - nT_s)\right)$$

$$= \begin{cases} y[n] & t = nT_s \\ \text{bandlimited interpolation} & \text{otherwise} \end{cases}$$

. . . where "bandlimited interpolation" means that $y[n]$'s spectrum for $-\pi < \omega < \pi$ is perfectly reconstructed by $y(t)$'s spectrum for $-\frac{F_s}{2} < f < \frac{F_s}{2}$, and that $y(t)$ has no spectrum outside that range.

# Ideal Bandlimited Sinc Interpolation of a Periodic Signal

Putting it all together, we have:

$$x(t) = \sum_{k=-N}^{N} X_k e^{j2\pi kF_0 t}$$

$$x[n] = \sum_{k=-N}^{N} X_k e^{jk\omega_0 n},$$

$$y[n] = \sum_{k=-N}^{N} Y_k e^{jk\omega_0 n},$$

$$y(t) = \sum_{k=-N}^{N} Y_k e^{j2\pi kF_0 n},$$

where $\omega_0 = \frac{2\pi F_0}{F_s}$, and $N = \lfloor \frac{F_s/2}{F_0} \rfloor$, and $Y_k = H(k\omega_0)X_k$.

# Quiz

Go to the course webpage, and try the quiz!

# Outline

# Rectangular Averager

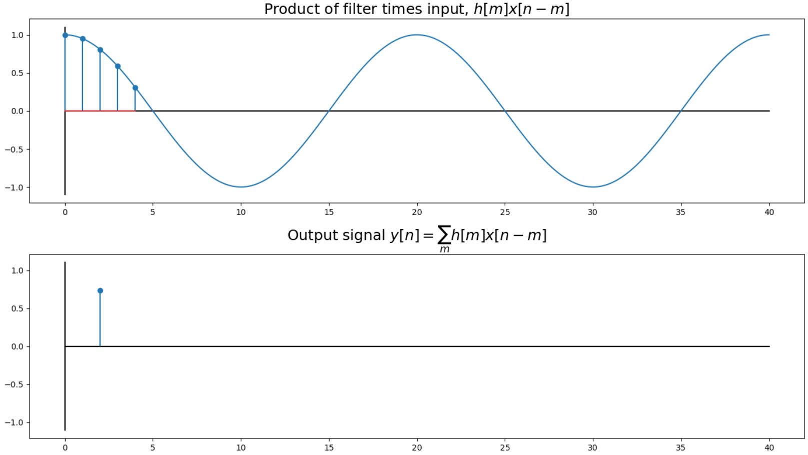
$$y[n] = \sum_m h[m]x[n-m]$$

Consider the case of the rectangular averager:

$$h[n] = \begin{cases} \frac{1}{N} & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

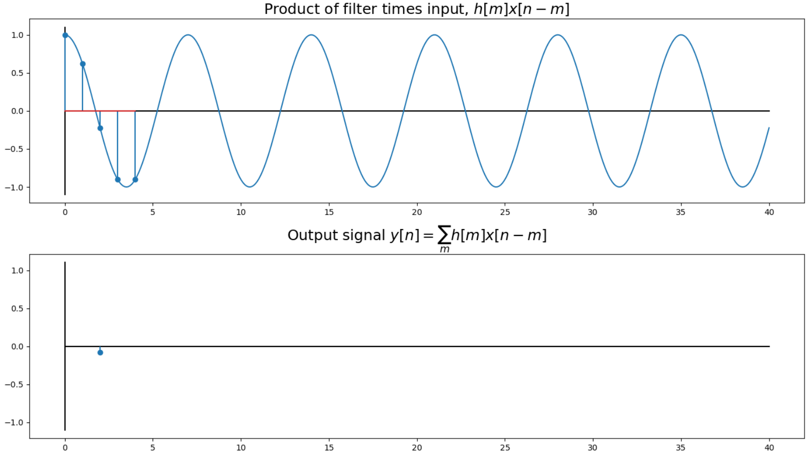
# Rectangular Averaging: Low-Frequency Cosine

When the input is low-frequency, the output of an averager is almost the same as the input:



# Rectangular Averaging: High-Frequency Cosine

When the input is high-frequency, the system averages over almost one complete period, so the output is close to zero:



# Rectangular Averaging: General Case

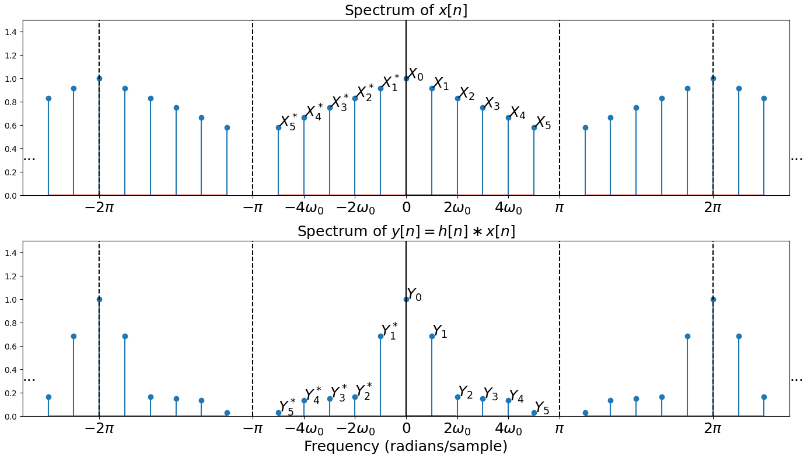
Remember the general form for the frequency response:

$$\begin{aligned} Y &= X \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} \\ &= \frac{X}{N} \sum_{m=-(N-1)/2}^{(N-1)/2} e^{-j\omega m} \\ &= \frac{X}{N} \left( 1 + 2 \sum_{m=1}^{(N-1)/2} \cos(\omega m) \right) \end{aligned}$$

- If $\omega$ is very small, all terms are positive, so the output is large.
- If $\omega$ is larger, then the summation includes both positive and negative terms, so the output is small.

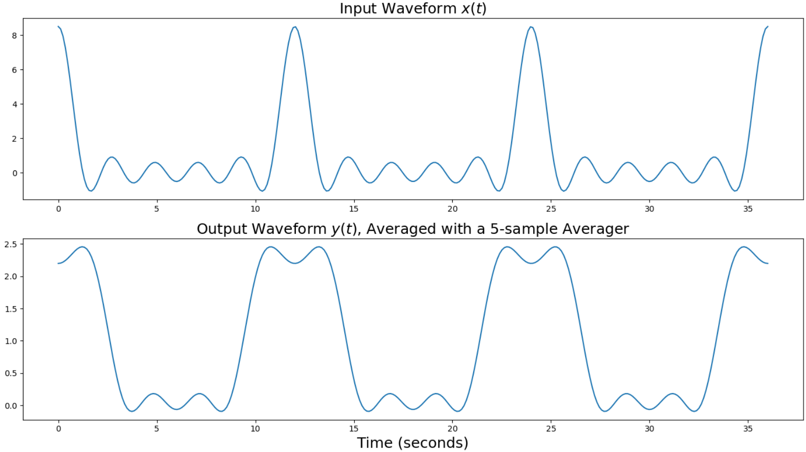
# Spectral Plots: Rectangular Averager

The averager retains low-frequency components, but reduces high-frequency components:



# Waveforms: Rectangular Averager

The averager tends to smooth out the waveform:



# Outline

# Binary Differencer

$$y[n] = \sum_m h[m]x[n-m]$$

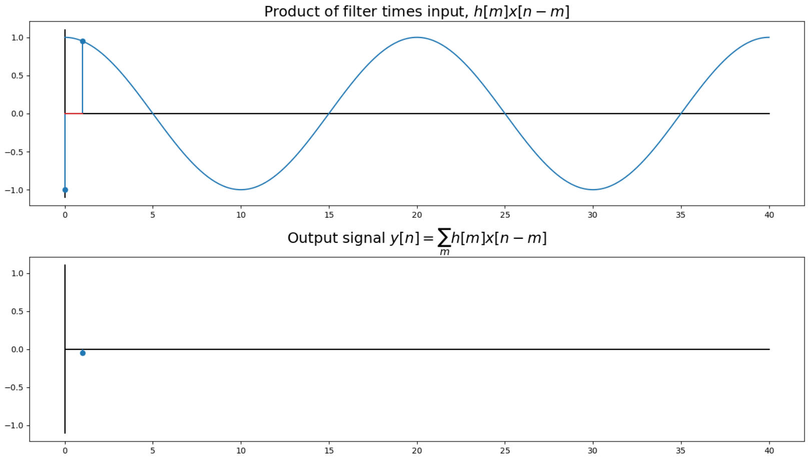
Consider the case of the binary differencer:

$$h[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

... so that $y[n] = x[n] - x[n-1]$.

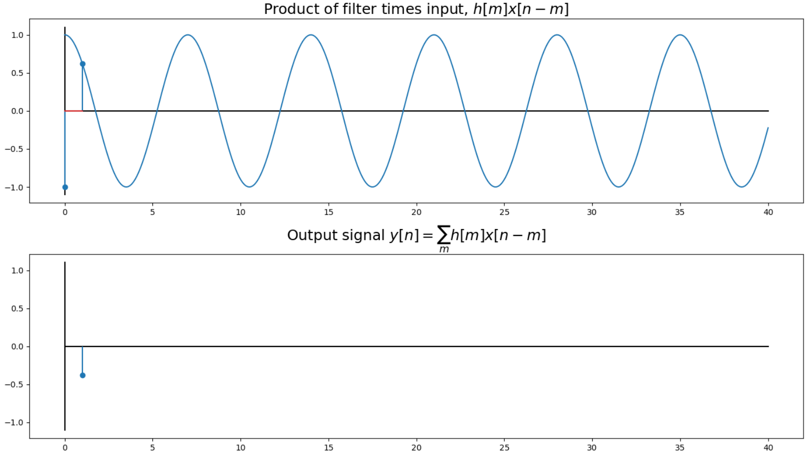
# Binary Differencer: Low-Frequency Cosine

When the input is low-frequency, the difference between neighboring samples is nearly zero:



# Binary Differencer: High-Frequency Cosine

When the input is high-frequency, the difference between neighboring samples is large:



# Binary Differencer: General Case

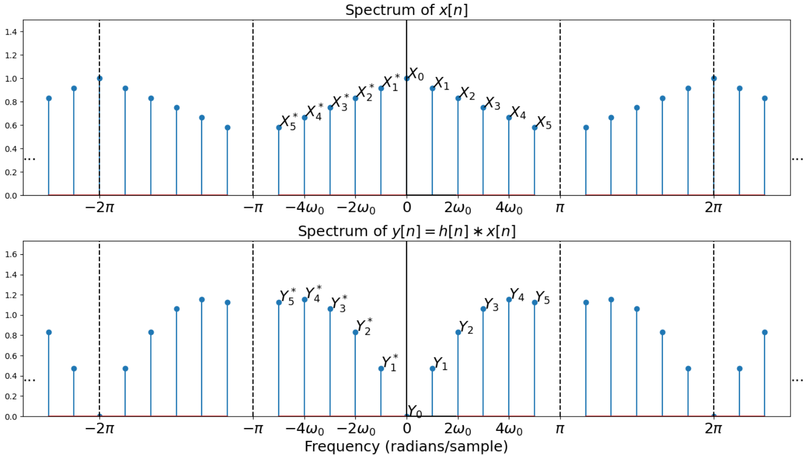
Remember the general form for the frequency response:

$$\begin{aligned} Y &= X \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} \\ &= X\left(1 - e^{-j\omega}\right) \\ &= X\left(e^{j\omega/2} - e^{-j\omega/2}\right) e^{-j\omega/2} \\ &= 2jX \sin\left(\frac{\omega}{2}\right) e^{-j\omega/2} \end{aligned}$$

- If $\omega$ is very small, $\sin(\omega/2)$ is very small
- As $\omega \to \pi$ (high frequencies), $\sin(\omega/2) \to 1$

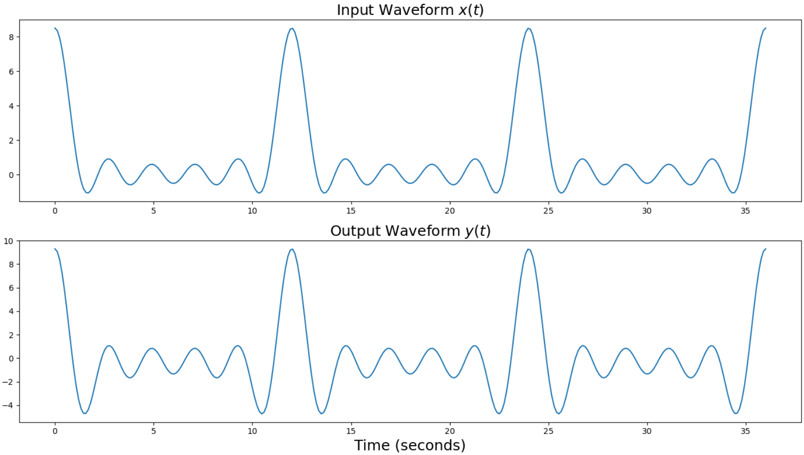
# Spectral Plots: Binary Differencer

The binary differencer removes the 0Hz component, but keeps high frequencies:



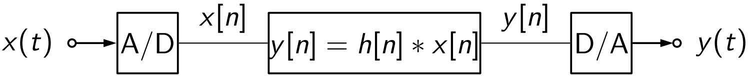
# Waveforms: Binary Differencer

The binary differencer removes the 0Hz component, and tends to emphasize “edges” in the waveform:



# Outline

# Conclusions




If $x(t)$ is periodic, then $y(t)$ is also periodic with the same period but different Fourier Series coefficients:

$$x(t) = \sum_{k=-N}^{N} X_k e^{j2\pi k F_0 t}, \qquad x[n] = \sum_{k=-N}^{N} X_k e^{jk\omega_0 n}$$

$$y[n] = \sum_{k=-N}^{N} Y_k e^{jk\omega_0 n}, \qquad y(t) = \sum_{k=-N}^{N} Y_k e^{j2\pi k F_0 t}$$

# Conclusions

The relationship between the Fourier series coefficients is given by the frequency response of the system:

$$Y = X \sum_m h[m] e^{-j\omega m}$$

- A rectangular averager is a low-pass filter: low-frequency signals pass through, but high-frequency signals are averaged out.
- A binary differencer is a high-pass filter: high-frequency signals pass through, but low-frequency signals are differenced out, especially the 0Hz component.