Sampling	Spectrum Plots	Oversampled	Undersampled	Sampling Theorem	DTFS	Summary

Lecture 9: Sampling Theorem

Mark Hasegawa-Johnson These slides are in the public domain.

ECE 401: Signal and Image Analysis

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How to sample a continuous-time signal

Suppose you have some continuous-time signal, x(t), and you'd like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$

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- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, $f < \frac{F_s}{2}$.
- If the Nyquist criterion is violated, then:
 - If $\frac{F_s}{2} < f < F_s$, then it will be aliased to

$$f_a = F_s - f$$
$$z_a = z^*$$

i.e., the sign of all sines will be reversed. • If $F_s < f < \frac{3F_s}{2}$, then it will be aliased to

$$f_a = f - F_s$$
$$z_a = z$$

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Spect	rum Plots					

The **spectrum plot** of a periodic signal is a plot with

- frequency on the X-axis,
- showing a vertical spike at each frequency component,
- each of which is labeled with the corresponding phasor.

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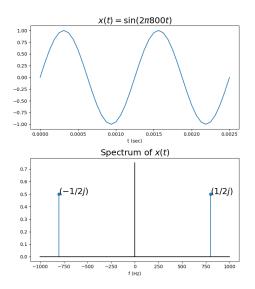
Sampling 000	Spectrum Plots	Oversampled 00000000	Undersampled 00000000	Sampling Theorem	DTFS 000000	Summary 0000
Examp	ole: Sine V	Vave				

$$\begin{aligned} x(t) &= \sin \left(2\pi 800t \right) \\ &= \frac{1}{2j} e^{j2\pi 800t} - \frac{1}{2j} e^{-j2\pi 800t} \end{aligned}$$

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The spectrum of x(t) is $\{(-800, -\frac{1}{2i}), (800, \frac{1}{2i})\}$.

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Exam	ple: Sine V	Vave				



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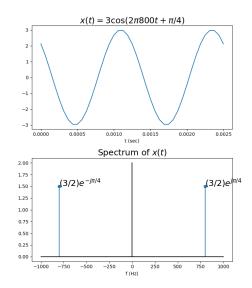
Example: Quadrature Cosine

$$\begin{aligned} x(t) &= 3\cos\left(2\pi 800t + \frac{\pi}{4}\right) \\ &= \frac{3}{2}e^{j\pi/4}e^{j2\pi 800t} + \frac{3}{2}e^{-j\pi/4}e^{-j2\pi 800t} \end{aligned}$$

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The spectrum of x(t) is $\{(-800, \frac{3}{2}e^{-j\pi/4}), (800, \frac{3}{2}e^{j\pi/4})\}.$

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Overs	ampled Sig	gnals				

A signal is called **oversampled** if $F_s > 2f$ (e.g., so that sinc interpolation can reconstruct it from its samples).

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The spectrum plot of a **discrete-time periodic signal** is a regular spectrum plot, but with the X-axis relabeled. Instead of frequency in Hertz= $\begin{bmatrix} cycles \\ second \end{bmatrix}$, we use

$$\omega \left[\frac{\text{radians}}{\text{sample}} \right] = \frac{2\pi \left[\frac{\text{radians}}{\text{cycle}} \right] f \left[\frac{\text{cycles}}{\text{second}} \right]}{F_s \left[\frac{\text{samples}}{\text{second}} \right]}$$

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Remember that a discrete-time signal has energy at

- f and -f, but also $F_s f$ and $-F_s + f$, and $F_s + f$ and $-F_s f$, and...
- ω and $-\omega$, but also $2\pi \omega$ and $-2\pi + \omega$, and $2\pi + \omega$ and $-2\pi \omega$, and . . .

Which ones should we plot? Answer: **plot all of them!** Usually we plot a few nearest the center, then add "…" at either end, to show that the plot continues forever.

Sampling	Spectrum Plots	Oversampled	Undersampled	Sampling Theorem	DTFS	Summary
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Exam	ple: Sine V	Vave				

Let's sample at $F_s = 8000$ samples/second.

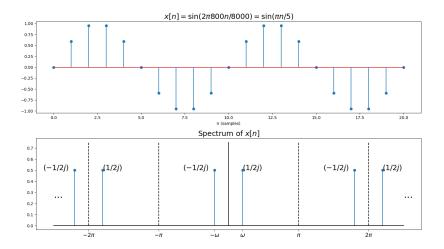
$$x[n] = \sin (2\pi 800n/8000)$$

= sin (\pi n/5)
= $\frac{1}{2j}e^{j\pi n/5} - \frac{1}{2j}e^{-j\pi n/5}$

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The spectrum of x[n] is $\{\ldots, (-\pi/5, -\frac{1}{2j}), (\pi/5, \frac{1}{2j}), \ldots\}$.

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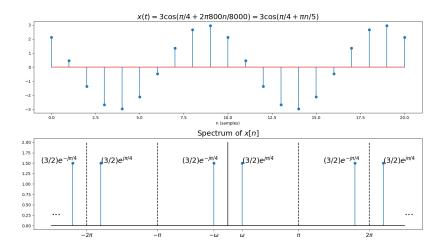
Example: Quadrature Cosine

$$\begin{aligned} x[n] &= 3\cos\left(2\pi 800n/8000 + \frac{\pi}{4}\right) \\ &= 3\cos\left(\pi n/5 + \frac{\pi}{4}\right) \\ &= \frac{3}{2}e^{j\pi/4}e^{j\pi n/5} + \frac{3}{2}e^{-j\pi/4}e^{-j\pi n/5} \end{aligned}$$

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The spectrum of x[n] is $\{\ldots, (-\pi/5, \frac{3}{2}e^{-j\pi/4}), (\pi/5, \frac{3}{2}e^{j\pi/4}), \ldots\}$.

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Under	sampled S	ignals				

A signal is called **undersampled** if $F_s < 2f$ (e.g., so that sinc interpolation can't reconstruct it from its samples).

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Remember that a discrete-time signal has energy at

- f and -f, but also $F_s f$ and $-F_s + f$, and $F_s + f$ and $-F_s f$, and...
- ω and $-\omega$, but also $2\pi \omega$ and $-2\pi + \omega$, and $2\pi + \omega$ and $-2\pi \omega$, and . . .

We still want to plot all of these, but now ω and $-\omega$ won't be the spikes closest to the center. Instead, some other spike will be closest to the center.

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Let's still sample at $F_s = 8000$, but we'll use a sine wave at f = 4800Hz, so it gets undersampled.

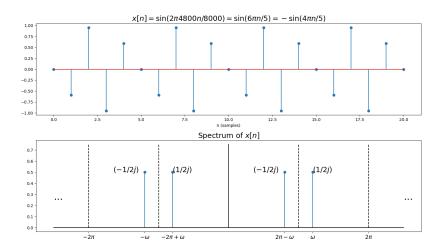
$$x[n] = \sin (2\pi 4800 n/8000)$$

= sin (6\pi n/5)
= - sin (4\pi n/5)
= -\frac{1}{2j}e^{j4\pi n/5} + \frac{1}{2j}e^{j4\pi n/5}

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The spectrum of x[n] is $\{\ldots, (-4\pi/5, \frac{1}{2j}), (4\pi/5, -\frac{1}{2j}), \ldots\}$.

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Example: Quadrature Cosine

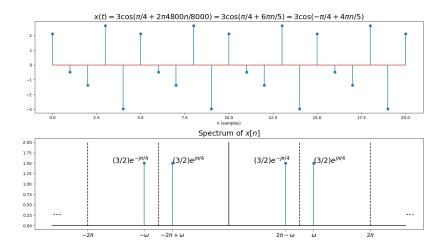
$$\begin{aligned} x[n] &= 3\cos\left(2\pi 4800n/8000 + \frac{\pi}{4}\right) \\ &= 3\cos\left(6\pi n/5 + \frac{\pi}{4}\right) \\ &= 3\cos\left(4\pi n/5 - \frac{\pi}{4}\right) \\ &= \frac{3}{2}e^{-j\pi/4}e^{j4\pi n/5} + \frac{3}{2}e^{j\pi/4}e^{-j4\pi n/5} \end{aligned}$$

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The spectrum of x[n] is $\{\dots, (-4\pi/5, \frac{3}{2}e^{j\pi/4}), (4\pi/5, \frac{3}{2}e^{-j\pi/4}), \dots\}.$

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Example: Quadrature Cosine



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Quiz						

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Go to the course web page, and try the quiz!

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General periodic continuous-time signals

Let's assume that x(t) is periodic with some period T_0 , therefore it has a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} = \sum_{k=0}^{\infty} 2|X_k| \cos\left(\frac{2\pi kt}{T_0} + \angle X_k\right)$$

We already know that $e^{j2\pi kt/T_0}$ will be aliased if $|k|/T_0 > F_N$. So let's assume that the signal is **band-limited:** it contains no frequency components with frequencies larger than $F_S/2$. That means that the only X_k with nonzero energy are the ones in the range $-\frac{N-1}{2} \le k \le \frac{N-1}{2}$, where $\frac{N-1}{2T_0} < \frac{F_s}{2}$:

$$x(t) = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kt/T_0}$$

Notice that, counting the k = 0 term, there are an odd number of harmonics (*N* is odd), in the range $-\frac{N-1}{2} \le k \le \frac{N-1}{2}$.

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Now let's sample that signal, at sampling frequency F_S :

$$x[n] = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kn/F_S T_0}$$
$$= \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{jk\omega_0 n},$$

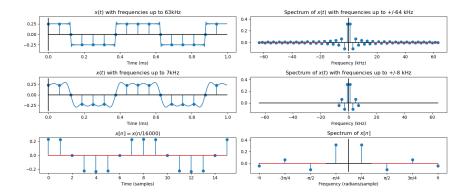
where the discrete-time fundamental frequency, expressed in radians/sample, is

$$\omega_0 = \frac{2\pi F_0}{F_s} = \frac{2\pi}{F_s T_0}$$

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Spectrum of a sampled periodic signal



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Sampling Spectrum Plots Oversampled Undersampled Sampling Theorem DTFS Summary 000000 The sampling theorem

As long as $-\pi \leq \omega_k \leq \pi$, we can recreate the continuous-time signal by either (1) using sinc interpolation, or (2) regenerating a continuous-time signal with the corresponding frequency:

$$f_k \left[\frac{\text{cycles}}{\text{second}} \right] = \frac{\omega_k \left[\frac{\text{radians}}{\text{sample}} \right] \times F_S \left[\frac{\text{samples}}{\text{second}} \right]}{2\pi \left[\frac{\text{radians}}{\text{cycle}} \right]}$$

$$x[n] = \cos(\omega_k n + \theta_k) \quad
ightarrow \quad x(t) = \cos(2\pi f_k t + \theta_k)$$

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The s	ampling th	leorem				

A continuous-time signal x(t) with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_S)$ if the samples are taken at a rate $F_s = 1/T_s$ that is $F_S \ge 2f_{max}$.

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Suppose we have a continuous-time periodic signal that is already band-limited, so its highest frequency is $\frac{N-1}{2T_0} < \frac{F_s}{2}$. Its continuous-time Fourier series is

$$x(t) = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kt/T_0}$$

Sampling Spectrum Plots Oversampled Undersampled Sampling Theorem DTFS Summary 000 0000000 Undersampled 0000000 0000000 0000000 0000000 Is it periodic in discrete time?

If the period T_0 is an integer number of samples ($T_0 = N/F_s$), then this signal is also periodic in discrete time:

$$x(t) = x(t + T_0)$$
$$x[n] = x[n + N]$$

If the signal is periodic in discrete time, then, by sampling its continuous-time Fourier series, we get its **discrete-time Fourier series**:

$$x[n] = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kn/F_5 T_0}$$

= $\sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j2\pi kn/N}, \qquad = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{jk\omega_0 n},$

where the discrete-time fundamental frequency, expressed in radians/sample, is

$$\omega_0 = \frac{2\pi F_0}{F_s} = \frac{2\pi}{F_s T_0} = \frac{2\pi}{N}$$



Remember that the Fourier series coefficients are computed as

$$X_k = rac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt$$

If the signal is periodic in discrete time (if T_0 is an integer number of samples), then we can compute exactly the same coefficients by averaging in discrete time:

$$X_{k} = \frac{1}{N} \sum_{0}^{N-1} x[n] e^{-j2\pi kn/N}$$



If x[n] is periodic with period N, then it has a Fourier series

$$x[n] = \sum_{k=-(N-1)/2}^{(N-1)/2} X_k e^{j\frac{2\pi kn}{N}},$$

whose coefficients can be computed as

$$X_k = \frac{1}{N} \sum_{0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

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Spectrum Plots						

The **spectrum plot** of a periodic signal is a plot with

- frequency on the X-axis,
- showing a vertical spike at each frequency component,
- each of which is labeled with the corresponding phasor.

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The spectrum plot of a **discrete-time periodic signal** is a regular spectrum plot, but with the X-axis relabeled. Instead of frequency in Hertz= $\begin{bmatrix} cycles \\ second \end{bmatrix}$, we use

$$\omega \left[\frac{\text{radians}}{\text{sample}} \right] = \frac{2\pi \left[\frac{\text{radians}}{\text{cycle}} \right] f \left[\frac{\text{cycles}}{\text{second}} \right]}{F_s \left[\frac{\text{samples}}{\text{second}} \right]}$$

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The sampling theorem								

A continuous-time signal x(t) with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_S)$ if the samples are taken at a rate $F_s = 1/T_s$ that is $F_S \ge 2f_{max}$.

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