Lecture 4: Fourier Synthesis

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ECE 401: Signal and Image Analysis

Outline

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\left(x(t)\right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Outline

Fourier's theorem

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0)=x(t)$$

can be written as

$$x(t)=\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kF_0 t}$$

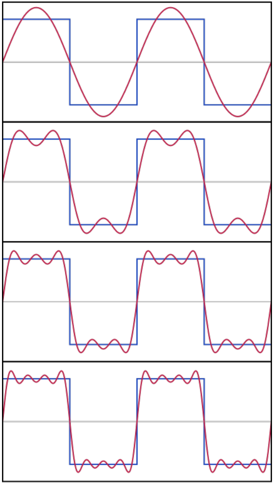
which is a special case of the spectrum for periodic signals: $f_k=kF_0$, and $a_k=X_k$, and

$$F_0=\frac{1}{T_0}$$

Analysis and Synthesis

- **Fourier Analysis** is the process of finding the spectrum, X_k, given the signal $x(t)$. I'll tell you how to do that next lecture.
- **Fourier Synthesis** is the process of generating the signal, $x(t)$, given its spectrum. I'll spend the rest of today's lecture showing examples and properties of synthesis.

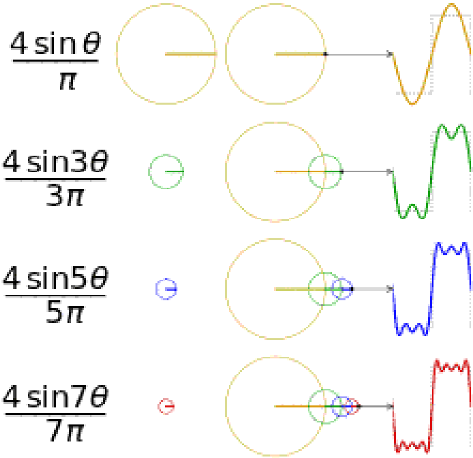
Example #1: Square wave



Jim.belk, Public domain image 2009, `https://commons.wikimedia.org/wiki/File:Fourier_Series.svg`

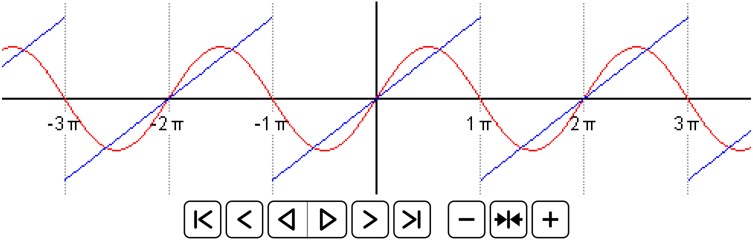
Example #1: Square wave

https://upload.wikimedia.org/wikipedia/commons/b/bd/Fourier_series_square_wave_circles_animation.svg



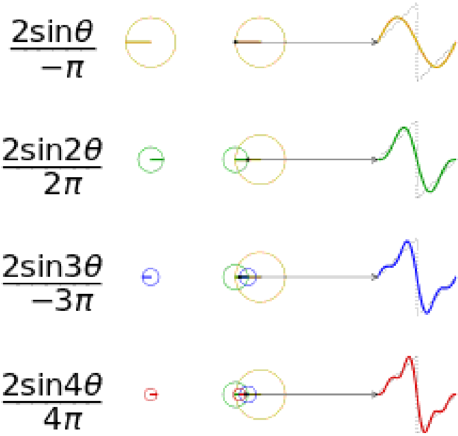
Example #2: Sawtooth wave

By Lucas Vieira, public domain 2009, https://commons.wikimedia.org/wiki/File:Periodic_identity_function.gif

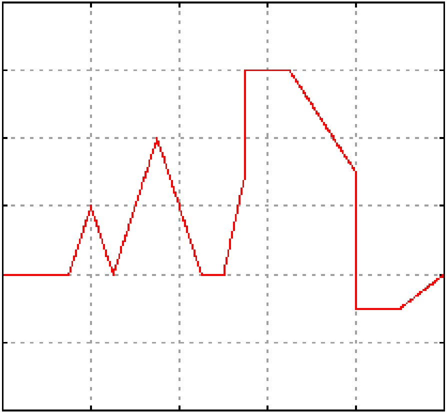


Example #2: Sawtooth wave

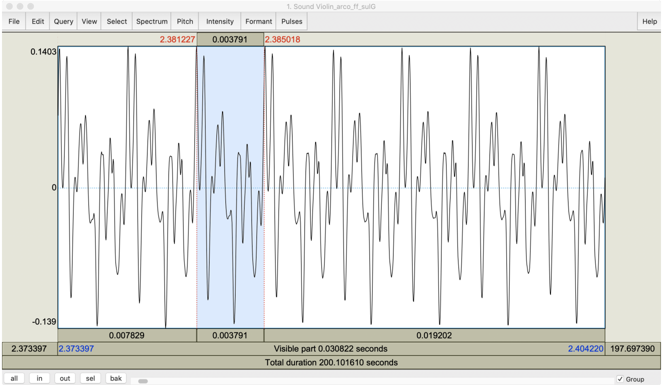
https://upload.wikimedia.org/wikipedia/commons/1/1e/Fourier_series_sawtooth_wave_circles_animation.svg



Example: A weird arbitrary signal

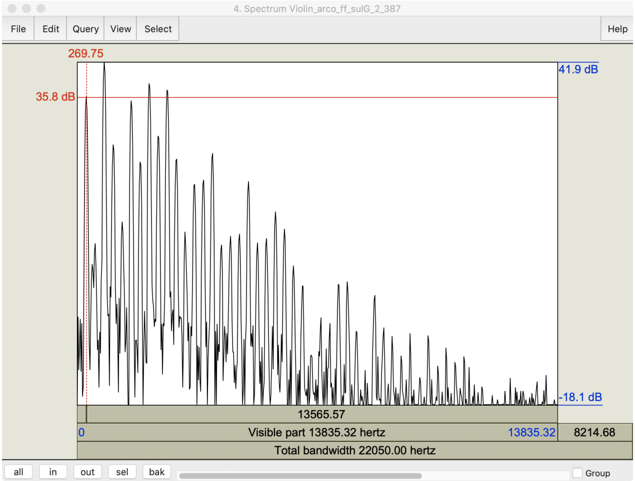


Example: Violin



Eight periods from the recording of a violin playing $f = 1/0.003791 = 262$Hz, i.e., C4 (middle C). Waveform distributed by **University of Iowa Electronic Music Studios**.

Example: Violin



Log magnitude spectrum ($20 \log_{10} |X_k|$) for the first 43 harmonics or so ($1 \leq k \leq 43$ or so) of a violin playing C4. Waveform distributed by **University of Iowa Electronic Music Studios**.

Outline

The missing fundamental

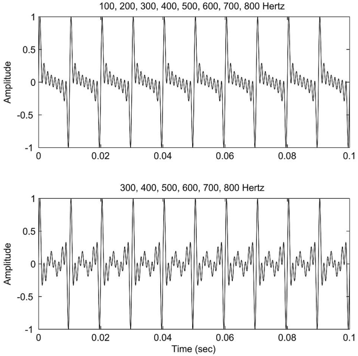
Suppose we have a signal of the form

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

Humans will hear the frequency F_0 to be the pitch of this sound, even if $X_{-1} = X_1 = 0$, i.e., even if the sound has no energy at the frequency F_0.

The missing fundamental

Our visual perception matches auditory perception. For example:



Quiz

Try the quiz! Go to the course webpage, and click on today's date.

The autocorrelation theory of pitch perception

The mystery of the missing fundamental was solved by Licklider in 1951. He showed that we can explain why humans hear F_0 as the pitch if we assume that humans use the following strategy:

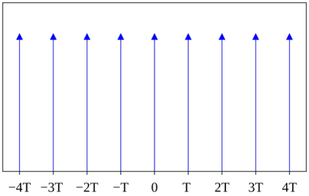
1. Your ear divides the sound into separate harmonics, each of which is tracked by a perceptual neuron.
2. The k^{th} perceptual neuron synapses onto a timer neuron in some way that removes the phase of the harmonic, i.e., it creates a signal $y(t)$ whose harmonic amplitudes are $Y_k = |X_k|$, the absolute value.
3. Finally, a third neuron adds together all of the zero-phase harmonics.

The autocorrelation theory of pitch perception

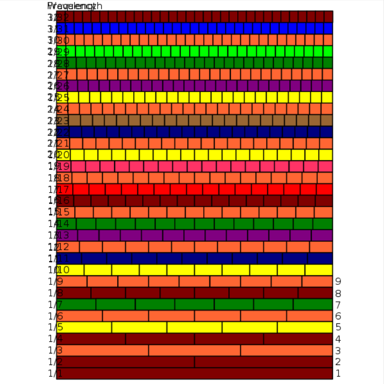
Suppose Y_k are all positive real numbers (zero phase). Then

$$\sum_{k=-K}^{K} Y_k e^{j2\pi kF_0 t} = Y_0 + \sum_{k=1}^{K} 2Y_k \cos\left(2\pi kF_0 t\right)$$

$$\xrightarrow[K\rightarrow\infty]{} \begin{cases} \infty & t = 0, T_0, 2T_0, \ldots \\ 0 & \text{otherwise} \end{cases}$$



The autocorrelation theory of pitch perception



Outline

Summary

- **Fourier's Theorem:** Any periodic waveform, $x(t+T_0)=x(t)$, can be synthesized as

$$x(t)=\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kF_0 t}$$

- **Autocorrelation theory of pitch perception:** If all of the Y_k are positive real numbers, then

$$\sum_{k=-K}^{K} Y_k e^{j2\pi kF_0 t} = Y_0 + \sum_{k=1}^{K} 2Y_k \cos\left(2\pi kF_0 t\right)$$

$$\underset{K\rightarrow\infty}{\rightarrow} \begin{cases} \infty & t=0, T_0, 2T_0, \ldots \\ 0 & \text{otherwise} \end{cases}$$