Lecture 3: Spectrum

Mark Hasegawa-Johnson

ECE 401: Signal and Image Analysis

Outline

Beat tones

When two pure tones at similar frequencies are added together, you hear the two tones “beating” against each other.

Beat tones demo

Beat tones and Trigonometric identities

Beat tones can be explained using this trigonometric identity:

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

Let's do the following variable substitution:

$$\begin{aligned} a+b &= 2\pi f_1 t \\ a-b &= 2\pi f_2 t \\ a &= 2\pi f_{ave} t \\ b &= 2\pi f_{beat} t \end{aligned}$$

where $f_{ave} = \frac{f_1+f_2}{2}$, and $f_{beat} = \frac{f_1-f_2}{2}$.

Beat tones and Trigonometric identities

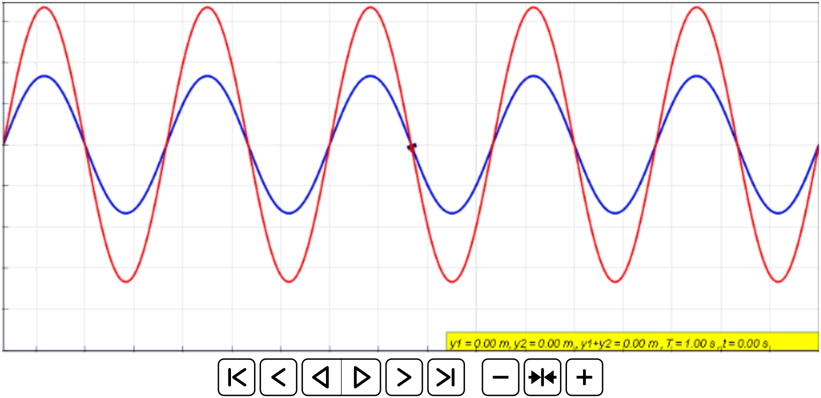
Re-writing the trigonometric identity, we get:

$$\frac{1}{2}\cos(2\pi f_1 t) + \frac{1}{2}\cos(2\pi f_2 t) = \cos(2\pi f_{beat} t)\cos(2\pi f_{ave} t)$$

So when we play two tones together, $f_1 = 110$Hz and $f_2 = 104$Hz, it sounds like we're playing a single tone at $f_{ave} = 107$Hz, multiplied by a beat frequency $f_{beat} = 3$ (double beats)/second.

Beat tones

by Adjwilley, CC-SA 3.0, https://commons.wikimedia.org/wiki/File:WaveInterference.gif



More complex beat tones

What happens if we add together, say, three tones?

$$\cos(2\pi 107t) + \cos(2\pi 110t) + \cos(2\pi 104t) = ???$$

For this, and other more complicated operations, it is much, much easier to work with complex exponentials, instead of cosines.

More complex beat tones

What happens if we add together, say, three tones?

$$x(t) = \cos(2\pi 107 t) + \cos(2\pi 110 t) + \cos(2\pi 104 t) = ???$$

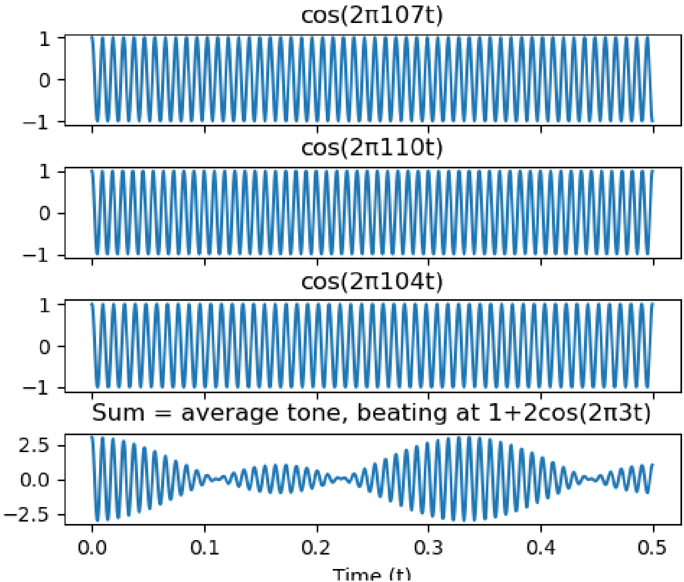
This is like a phasor example, except that all of the tones are at different frequencies.

$$\begin{aligned} x(t) &= \Re\left\{ e^{j2\pi 107 t} + e^{j2\pi 110 t} + e^{j2\pi 104 t} \right\} \\ &= \Re\left\{ \left(1 + e^{j2\pi 3t} + e^{-j2\pi 3t}\right) e^{j2\pi 107 t} \right\} \end{aligned}$$

So we just have to do this phasor addition:

$$1 + e^{j2\pi 3t} + e^{-j2\pi 3t} = 1 + 2\cos\left(2\pi 3 t\right)$$

Triple-beat example



Outline

Phasor representation of a general sum of sinusoids

In general, if $x(t)$ is a sum of sines and cosines,

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(2\pi f_k t + \theta_k\right)$$

Then it has a phasor notation

$$x(t) = A_0 + \sum_{k=1}^{N} \Re\left\{A_k e^{j\theta_k} e^{j2\pi f_k t}\right\}$$

Two-sided spectrum

The $\Re\{z\}$ operator is annoying. In order to get rid of it, let's take advantage of Euler's formula $\Re\{z\} = \frac{1}{2}(z + z^*)$ to write:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(2\pi f_k t + \theta_k\right)$$
$$= \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

In order to do that, we need to define a_k like this:

$$a_k = \begin{cases} A_0 & k = 0 \\ \frac{1}{2} A_k e^{j\theta_k} & k > 0 \\ \frac{1}{2} A_{-k} e^{-j\theta_{-k}} & k < 0 \end{cases}$$

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\left(x(t)\right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Quiz

Try the quiz! Go to the course webpage, and click on today's date to try the quiz.

Outline

Properties of a spectrum

Spectrum representation is nice to use because

- It's so general. Any signal made up of pure tones can be written this way.
- Many signal processing operations can be written directly in the spectral domain (as operations on a_k), without converting back to $x(t)$.

Property #1: Scaling

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Suppose we scale it by a factor of G:

$$y(t) = Gx(t)$$

That just means that we scale each of the coefficients by G:

$$y(t) = \sum_{k=-N}^{N} (Ga_k)\, e^{j2\pi f_k t}$$

Property #2: Adding a constant

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Suppose we add a constant, C:

$$y(t) = x(t) + C$$

That just means that we add that constant to a_0:

$$y(t) = (a_0 + C) + \sum_{k\neq 0} a_k e^{j2\pi f_k t}$$

Property #3: Adding two signals

Suppose we have two signals:

$$x(t) = \sum_{n=-N}^{N} a'_n e^{j2\pi f'_n t}$$

$$y(t) = \sum_{m=-M}^{M} a''_m e^{j2\pi f''_m t}$$

and we add them together:

$$z(t) = x(t) + y(t) = \sum_k a_k e^{j2\pi f_k t}$$

where, if a frequency f_k comes from both $x(t)$ and $y(t)$, then we have to do phasor addition:

$$\text{If } f_k = f'_n = f''_m \text{ then } a_k = a'_n + a''_m$$

Property #4: Time shift

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to time shift it by τ seconds:

$$y(t) = x(t - \tau)$$

Time shift corresponds to a **phase shift** of each spectral component:

$$y(t) = \sum_{k=-N}^{N} \left(a_k e^{-j2\pi f_k \tau} \right) e^{j2\pi f_k t}$$

Property #5: Frequency shift

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to shift it in frequency by some constant overall shift, F:

$$y(t) = \sum_{k=-N}^{N} a_k e^{j2\pi (f_k+F) t}$$

Frequency shift corresponds to amplitude modulation (multiplying it by a complex exponential at the carrier frequency F):

$$y(t) = x(t) e^{j2\pi F t}$$

Property #6: Differentiation

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to differentiate it:

$$y(t) \propto \frac{dx}{dt}$$

Differentiation corresponds to scaling each spectral coefficient by $j2\pi f_k$:

$$y(t) = \sum_{k=-N}^{N} \left(j2\pi f_k a_k\right) e^{j2\pi f_k t}$$

Outline

Representing beat tones with no trig identities

One nice thing about the spectral representation is that you can analyze beat tones without remembering any trig identities.
First, write out the spectrum:

$$2\cos(2\pi f_1 t) + 2\cos(2\pi f_2 t) = e^{-j2\pi f_2 t} + e^{-j2\pi f_1 t} + e^{j2\pi f_1 t} + e^{j2\pi f_2 t}$$

Representing beat tones with no trig identities

Second, write the spectrum in terms of the carrier frequency $f_c = \frac{f_2+f_1}{2}$ and the beat frequency $f_b = \frac{f_2-f_1}{2}$:

$$e^{-j2\pi f_2 t}+e^{-j2\pi f_1 t}+e^{j2\pi f_1 t}+e^{j2\pi f_2 t} = e^{-j2\pi(f_c+b_b)t}+e^{-j2\pi(f_c-f_b)t}+e^{j2\pi(f_c-f_b$$

Representing beat tones with no trig identities

Third, remember that $xy + xz = x(y + z)$, so

$$e^{-j2\pi(f_c+b_b)t}+e^{-j2\pi(f_c-f_b)t}+e^{j2\pi(f_c-f_b)t}+e^{j2\pi(f_c+f_b)t} = \left(e^{-j2\pi f_b t} + e^{j2\pi f_b t}\right)$$

Representing beat tones with no trig identities

Solving for beat tones using the spectrum took longer than using a trig identity, but it has the key advantage that we didn't have to remember any trig identity. The result is the same, but we can do it even when we don't have an internet connection.

$$2\cos(2\pi f_1 t) + 2\cos(2\pi f_2 t) = 4\cos(2\pi f_b t)\cos(2\pi f_c t)$$

Outline

Summary

- **Spectrum:** The spectrum of any sum of cosines is the set of complex-valued spectral coefficients, a_k, matched with the frequencies f_k, such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

- **Properties of the spectrum:** signal processing operations that can be done directly in the spectrum, without first recomputing the waveform, include scaling, adding, time shift, frequency shift, and differentiation.