UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
Fall 2024

EXAM 3

Wednesday, December 18, 2023, 7:00-10:00pm

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave unevaluated summations or integrals. Beyond that point, you need not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 7:00pm; you will need to photograph and upload your answers by exactly 10:00pm.
- There will be a total of 200 points in the exam (9 problems). Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

netid: ______________________________

Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Spectrum

$$\textbf{Scaling:}\ \ y(t)=Gx(t)=\sum_{k=-N}^{N}\left(Ga_k\right)e^{j2\pi f_k t}$$

$$\textbf{Add a Constant:}\ \ y(t)=x(t)+C=(a_0+C)+\sum_{k\neq 0}a_k e^{j2\pi f_k t}$$

$$\textbf{Add Signals:}\ \ \text{If } f_k=f'_n=f''_m \ \text{ then } \ a_k=a'_n+a''_m$$

$$\textbf{Time Shift:}\ \ y(t)=x(t-\tau)=\sum_{k=-N}^{N}\left(a_k e^{-j2\pi f_k\tau}\right)e^{j2\pi f_k t}$$

$$\textbf{Frequency Shift:}\ \ y(t)=x(t)e^{j2\pi Ft}=\sum_{k=-N}^{N}a_k e^{j2\pi(f_k+F)t}$$

$$\textbf{Differentiation:}\ \ y(t)=\frac{dx}{dt}=\sum_{k=-N}^{N}\left(j2\pi f_k a_k\right)e^{j2\pi f_k t}$$

Fourier Series

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases} z & f \bmod F_s<-f \bmod F_s\\ z^* & f \bmod F_s>-f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

Convolution

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

Linearity, Shift-Invariance, Causality, Stability

1. **Linearity:** $x_1[n]+x_2[n]\rightarrow y_1[n]+y_2[n]$
2. **Shift-Invariance:** $x_1[n-n_0]\rightarrow y_1[n-n_0]$
3. **Causality:** $h[n]=0$ for $n<0$
4. **Stability:** $|x[n]|<M_1\Rightarrow|y[n]|<M_2$

Frequency Response and DTFT

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$
$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega$$
$$h[n] * \cos(\omega n) = |H(\omega)| \cos\left(\omega n + \angle H(\omega)\right)$$

DTFT Properties

1. Periodicity: $X(\omega + 2\pi) = X(\omega)$
2. Linearity: $z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$
3. Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
4. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
5. Convolution in Time: $y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$
6. Convolution in Frequenty: $y[n] = w[n]x[n] \leftrightarrow Y(\omega) = \frac{1}{2\pi} W(\omega) * X(\omega)$

Rectangular & Hamming Windows; Ideal LPF

$$w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = e^{-\frac{j\omega(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$
$$w_H[n] = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) w_R[n] \leftrightarrow W_H(\omega) = 0.54 W_R(\omega) - 0.23 W_R\left(\omega - \frac{2\pi}{N-1}\right) - 0.23 W_R\left(\omega + \frac{2\pi}{N-1}\right)$$
$$h_{\text{ideal}}[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Discrete Fourier Transform

$$\textbf{Analysis:} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$
$$\textbf{Synthesis:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$
$$\textbf{Circular Convolution:} \quad x[n] \circledast h[n] = \sum_{m=0}^{N-1} h[m]\, x\left[\langle n - m \rangle_N\right]$$

Overlap-Add

$$N \ge M + L - 1$$
$$x_t[n] = x[n + tM] w[n]$$
$$Y_t[k] = X_t[k] H[k]$$
$$y[n] = \sum_t y_t[n - tM]$$

Z Transform Pairs

$$
\begin{aligned}
b_k z^{-k} &\leftrightarrow b_k \delta[n-k] \\
\frac{1}{1-az^{-1}} &\leftrightarrow a^n u[n] \\
\frac{1}{(1-e^{-\sigma_1-j\omega_1}z^{-1})(1-e^{-\sigma_1+j\omega_1}z^{-1})} &\leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin\left(\omega_1(n+1)\right) u[n]
\end{aligned}
$$

Notch Filter

$$
\begin{aligned}
H(z) &= \frac{(1-e^{j\omega_1}z^{-1})(1-e^{-j\omega_1}z^{-1})}{(1-ae^{j\omega_1}z^{-1})(1-ae^{-j\omega_1}z^{-1})} \\
\mathrm{BW} &= -2\ln(a)
\end{aligned}
$$