UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
Fall 2024

EXAM 3

Wednesday, December 18, 2023, 7:00-10:00pm

- This is a CLOSED BOOK exam.
- You are permitted two sheets of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave unevaluated summations or integrals. Beyond that point, you need not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 7:00pm; you will need to photograph and upload your answers by exactly 10:00pm.
- There will be a total of 200 points in the exam (9 problems). Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

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Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Spectrum

$$\textbf{Scaling:}\ \ y(t)=Gx(t)=\sum_{k=-N}^{N}\left(Ga_k\right)e^{j2\pi f_k t}$$

$$\textbf{Add a Constant:}\ \ y(t)=x(t)+C=(a_0+C)+\sum_{k\neq 0}a_k e^{j2\pi f_k t}$$

$$\textbf{Add Signals:}\ \ \text{If } f_k=f'_n=f''_m \ \ \text{then} \ \ a_k=a'_n+a''_m$$

$$\textbf{Time Shift:}\ \ y(t)=x(t-\tau)=\sum_{k=-N}^{N}\left(a_k e^{-j2\pi f_k\tau}\right)e^{j2\pi f_k t}$$

$$\textbf{Frequency Shift:}\ \ y(t)=x(t)e^{j2\pi Ft}=\sum_{k=-N}^{N}a_k e^{j2\pi(f_k+F)t}$$

$$\textbf{Differentiation:}\ \ y(t)=\frac{dx}{dt}=\sum_{k=-N}^{N}\left(j2\pi f_k a_k\right)e^{j2\pi f_k t}$$

Fourier Series

$$\textbf{Analysis:}\ \ X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\ \ x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases} z & f \bmod F_s<-f \bmod F_s \\ z^* & f \bmod F_s>-f \bmod F_s \end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

Convolution

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

Linearity, Shift-Invariance, Causality, Stability

1. **Linearity:** $x_1[n]+x_2[n]\rightarrow y_1[n]+y_2[n]$
2. **Shift-Invariance:** $x_1[n-n_0]\rightarrow y_1[n-n_0]$
3. **Causality:** $h[n]=0$ for $n<0$
4. **Stability:** $|x[n]|<M_1\Rightarrow|y[n]|<M_2$

Frequency Response and DTFT

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$
$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega$$
$$h[n] * \cos(\omega n) = |H(\omega)| \cos\left(\omega n + \angle H(\omega)\right)$$

DTFT Properties

1. Periodicity: $X(\omega + 2\pi) = X(\omega)$
2. Linearity: $z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$
3. Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
4. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
5. Convolution in Time: $y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$
6. Convolution in Frequenty: $y[n] = w[n]x[n] \leftrightarrow Y(\omega) = \frac{1}{2\pi} W(\omega) * X(\omega)$

Rectangular & Hamming Windows; Ideal LPF

$$w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = e^{-\frac{j\omega(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$
$$w_H[n] = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) w_R[n] \leftrightarrow W_H(\omega) = 0.54 W_R(\omega) - 0.23 W_R\left(\omega - \frac{2\pi}{N-1}\right) - 0.23 W_R\left(\omega + \frac{2\pi}{N-1}\right)$$
$$h_{\text{ideal}}[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Discrete Fourier Transform

$$\textbf{Analysis:} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$
$$\textbf{Synthesis:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$
$$\textbf{Circular Convolution:} \quad x[n] \circledast h[n] = \sum_{m=0}^{N-1} h[m]\, x\left[\langle n - m \rangle_N\right]$$

Overlap-Add

$$N \ge M + L - 1$$
$$x_t[n] = x[n + tM] w[n]$$
$$Y_t[k] = X_t[k] H[k]$$
$$y[n] = \sum_t y_t[n - tM]$$

Z Transform Pairs

$$
\begin{aligned}
b_k z^{-k} &\leftrightarrow b_k \delta[n-k] \\
\frac{1}{1-az^{-1}} &\leftrightarrow a^n u[n] \\
\frac{1}{(1-e^{-\sigma_1 - j\omega_1} z^{-1})(1-e^{-\sigma_1 + j\omega_1} z^{-1})} &\leftrightarrow \frac{1}{\sin(\omega_1)} e^{-\sigma_1 n} \sin\left(\omega_1 (n+1)\right) u[n]
\end{aligned}
$$

Notch Filter

$$
\begin{aligned}
H(z) &= \frac{(1-e^{j\omega_1} z^{-1})(1-e^{-j\omega_1} z^{-1})}{(1-ae^{j\omega_1} z^{-1})(1-ae^{-j\omega_1} z^{-1})} \\
\mathrm{BW} &= -2\ln(a)
\end{aligned}
$$

1. (23 points) Consider a periodic signal $x(t)$, with period T_0, and values:

$$x(t) = \begin{cases} t & 0 \leq t < \frac{T_0}{2} \\ T_0 - t & \frac{T_0}{2} \leq t < T_0 \end{cases}$$

The constant part of this signal is $X_0 = \frac{T_0}{4}$. The easiest way to find the other Fourier series coefficients is to notice that $y(t) = \frac{dx}{dt}$, where

$$y(t) = \begin{cases} 1 & 0 \leq t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \leq t < T_0 \end{cases} \leftrightarrow Y_k = \begin{cases} 0 & k \text{ even} \\ -\frac{2j}{\pi k} & k \text{ odd} \end{cases}$$

Find X_k, the Fourier series coefficients of $x(t)$, for all $k \neq 0$.

Solution:

$$y(t) = \frac{dx}{dt} \leftrightarrow Y_k = \frac{j2\pi k}{T_0} X_k$$

Therefore

$$X_k = \begin{cases} 0 & k \text{ even} \\ -\frac{T_0}{(\pi k)^2} & k \text{ odd} \end{cases}$$

2. (22 points) Suppose that

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$
$$z[n] = y[n] - y[n-10]$$
$$H(\omega) = \frac{Z(\omega)}{X(\omega)}$$

Find $H(\omega)$.

Solution:

$$h[n] = u[n] - u[n-10]$$

Therefore

$$H(\omega) = e^{-j\frac{9\omega}{2}} \frac{\sin(5\omega)}{\sin(\omega/2)}$$

3. (23 points) Consider an unstable linear time-invariant system whose impulse response is

$$h[n] = \begin{cases} 0 & n \leq 0 \\ n & n \geq 0 \end{cases}$$

Suppose $x[n] = u[n]$, the unit step function. What is $y[n] = h[n] * x[n]$? Hint: The details of this answer are a little tricky. Check some special cases, e.g., $y[1]$, $y[2]$, and $y[3]$, to make sure you have the details right.

Solution:

$$\begin{aligned} y[n] &= \sum_{m=0}^{n} m \\ &= \frac{1}{2}n(n+1) \end{aligned}$$

4. (22 points) Suppose $x[n] = \delta[n-3] - \delta[n-17]$, where $\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$. Find $X[k]$, the length-64 DFT of $x[n]$.

Solution:

$$X[k] = e^{-j\frac{2\pi 3k}{64}} - e^{-j\frac{2\pi 17k}{64}}$$

5. (22 points) Suppose $x[n] = \delta[n-3] - \delta[n-17]$, where $\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$, and suppose that

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^2 & 0 \le n \le 63 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $Y[k] = H[k]X[k]$, where $X[k]$, $H[k]$, and $Y[k]$ are the length-64 discrete Fourier transforms of $x[n]$, $h[n]$, and $y[n]$ respectively. Find $y[n]$.

Solution: The solution is the circular convolution of $x[n]$ and $h[n]$. The linear convolution is

$$h[n] * x[n] = \begin{cases} 0 & 0 \le n < 3 \\ \frac{1}{4} & 3 \le n < 17 \\ 0 & 17 \le n < 67 \\ -\frac{1}{4} & 67 \le n < 81 \\ 0 & 81 \le n \end{cases}$$

After time-domain aliasing with a period of 64, the result is

$$h[n] \circledast x[n] = 0$$

6. (22 points) Suppose $h[n]$ is nonzero only in the range $0 \leq n \leq 64$, i.e., it is 65 samples long. Suppose that $x[n]$ is an infinite-length signal divided into length-64 frames, thus

$$x_t[n] = \begin{cases} x[n+64t] & 0 \leq n \leq 63, \quad -\infty < t < \infty \\ 0 & \text{otherwise} \end{cases}$$

Suppose $Y_t[k] = H[k]X_t[k]$, where $X_t[k]$, $H[k]$, and $Y_t[k]$ are the length-128 discrete Fourier transforms of $x_t[n]$, $h[n]$, and $y_t[n]$, respectively. Suppose that there is some particular $x[n]$ and $h[n]$ such that all of the $y_t[n]$ turn out exactly the same: suppose that they all turn out to be

$$y_t[n] = \begin{cases} \left(\frac{1}{2}\right)^n & 0 \leq n \leq 127, \quad -\infty < t < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define $y[n] = h[n] * x[n]$, the linear convolution of $h[n]$ with $x[n]$. Find the numerical value of $y[n]$. Hint: You may find it useful to write your answer in terms of $m(n) = 64\lfloor \frac{n}{64} \rfloor$, which is the smallest multiple of 64 such that $m(n) \leq n$.

Solution: If we overlap and add the frames, we get

$$y[n] = \sum_{t=-\infty}^{\infty} y_t[n - 64t]$$

Taking the hint from the problem statement, and noticing that the frames only overlap by half, we get that

$$y[n] = \left(\frac{1}{2}\right)^{n-m(n)} + \left(\frac{1}{2}\right)^{64+n-m(n)}$$

7. (22 points) Consider the system

$$y[n] = x[n] + \frac{2}{3}x[n-1] + \frac{2}{3}y[n-1]$$

Plot a pole-zero plot for the system function $H(z) = Y(z)/X(z)$, showing the locations in the complex plane of every zero and every pole of $H(z)$.

Solution: The plot should show a complex plane (real and imaginary axes), with a pole at $z = \frac{2}{3}$, and a zero at $z = -\frac{2}{3}$.

8. (22 points) You have a signal sampled at $F_s = 10,000$ samples/second, corrupted by an annoying tone at 1000Hz. You would like to get rid of the 1000Hz tone, while leaving all components of the signal at frequencies greater than 1010Hz or less than 990Hz essentially unchanged. If you choose appropriate constants a, b, c, and d, you can accomplish this denoising task using the following system:

$$y[n] = x[n] + ax[n-1] + bx[n-2] + cy[n-1] + dy[n-2]$$

Choose constants a, b, c, and d such that the system above will get rid of the 1000Hz tone without much changing input components at frequencies above 1010Hz or below 990Hz.

Solution: The center frequency and bandwidth are

$$\begin{aligned} \omega_c &= \frac{2\pi 1000}{10000} = 0.2\pi \\ -2\ln(a) &= \frac{2\pi 20}{10000} = 0.004\pi \end{aligned}$$

So the coefficients are

$$\begin{aligned} a &= -2\cos(0.2\pi) \\ b &= 1 \\ c &= 2e^{-0.002\pi}\cos(0.2\pi) \\ d &= -e^{-0.004\pi} \end{aligned}$$

9. (22 points) Suppose you have a second-order system whose system function is

$$H(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})}$$

You would like to find the impulse response $h[n]$ of this system. One way to do so is by finding coefficients c and d such that

$$H(z) = \frac{c}{1 - az^{-1}} + \frac{d}{1 - bz^{-1}} \leftrightarrow h[n] = (ca^n + db^n)u[n]$$

Find c and d in terms of a and b.

Solution:

$$c = \frac{1}{1 - b/a}$$
$$d = \frac{1}{1 - a/b}$$

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