UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2024

EXAM 2

Monday, November 11, 2024

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave any unresolved integrals or summations in your answer. Beyond that point, you do not need to simplify your answer unless simplification is explicitly requested.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

Convolution

$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Linearity, Shift-Invariance, Causality, Stability

a) **Linearity:** $x_1[n] + x_2[n] \to y_1[n] + y_2[n]$

b) **Shift-Invariance:** $x_1[n-n_0] \to y_1[n-n_0]$

c) **Causality:** $h[n] = 0$ for $n < 0$

d) **Stability:** $|x[n]| < M_1 \Rightarrow |y[n]| < M_2$

Frequency Response

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$h[n] * \cos(\omega n) = |H(\omega)| \cos(\omega n + \angle H(\omega))$$

DTFT Properties

a) Periodicity: $X(\omega + 2\pi) = X(\omega)$

b) Linearity: $z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$

c) Time Shift: $x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$

d) Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$

e) Convolution in Time: $y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$

f) Convolution in Frequenty: $y[n] = w[n]x[n] \leftrightarrow Y(\omega) = \frac{1}{2\pi} W(\omega) * X(\omega)$

Rectangular Window and Ideal LPF

$$\sum_{n=0}^{L-1} a^n = \frac{1-a^L}{1-a}$$

$$w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = e^{-\frac{j\omega(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$h_{\text{ideal}}[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Hamming Window

$$w_H[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$W_H(\omega) = 0.54 W_R(\omega) + 0.23 W_R\left(\omega - \frac{2\pi}{N-1}\right) + 0.23 W_R\left(\omega + \frac{2\pi}{N-1}\right)$$

1. (28 points) Consider a system whose output, $y[n]$, is an average of the samples of its input:

$$y[n] = \sum_{m=m_1}^{m_2} x[n-m]$$

The behavior of the system depends on the values of m_1 and m_2. You may assume that $m_1 < m_2$, but you should consider all possibilities in the range $-\infty \leq m_1 < m_2 \leq \infty$.

a) Are there any values of m_1 or m_2 for which this system is nonlinear? If so, list all values of m_1 and m_2 for which the system is nonlinear. If not, prove that the system is linear.

Solution: It is linear for all values of m_1 and m_2. Proof:

$$y_1[n] = \sum_{m=m_1}^{m_2} x_1[n-m]$$

$$y_2[n] = \sum_{m=m_1}^{m_2} x_2[n-m]$$

$$\begin{aligned} x[n] = x_1[n] + x_2[n] \rightarrow y[n] &= \sum_{m=m_1}^{m_2} x[n-m] \\ &= \sum_{m=m_1}^{m_2} x_1[n-m] + x_2[n-m] \\ &= \sum_{m=m_1}^{m_2} x_1[n-m] + \sum_{m=m_1}^{m_2} x_2[n-m] \\ &= y_1[n] + y_2[n] \end{aligned}$$

b) Are there any values of m_1 or m_2 for which this system is shift-varying? If so, list all values of m_1 and m_2 for which the system is shift-varying. If not, prove that the system is shift-invariant.

Solution: It is shift-invariant for all values of m_1 and m_2. Proof:

$$y_1[n] = \sum_{m=m_1}^{m_2} x_1[n-m]$$

$$\begin{aligned} x[n] = x_1[n-n_0] \rightarrow y[n] &= \sum_{m=m_1}^{m_2} x[n-m] \\ &= \sum_{m=m_1}^{m_2} x_1[n-n_0-m] \\ &= y_1[n-n_0] \end{aligned}$$

c) Are there any values of m_1 or m_2 for which this system is non-causal? If so, list all values of m_1 and m_2 for which the system is non-causal. If not, prove that the system is causal.

Solution: It is non-causal if $m_1 < 0$, because

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

$$h[m] = \begin{cases} 1 & m_1 \le m \le m_2 \\ 0 & \text{otherwise} \end{cases}$$

...which is right-sided only if $m_1 \ge 0$.

d) Are there any values of m_1 or m_2 for which this system is unstable? If so, list all values of m_1 and m_2 for which the system is unstable. If not, prove that the system is stable.

Solution: 2cm It is stable only if both m_1 and m_2 are finite. If either $m_1 = -\infty$ or $m_2 = \infty$, then bounded inputs can give unbounded outputs, e.g., for $x[n] = 1$ we would have

$$y[n] = \sum_{m=m_1}^{m_2} 1 = \infty$$

2. (24 points) Suppose the input to a system is $x[n] = u[n] - u[n-L]$, and the impulse response is $h[n] = a^n u[n]$, where $u[n]$ is the unit step function ($u[n] = 1$ for $n \geq 0$, $u[n] = 0$ for $n < 0$). Find $x[n] * h[n]$ for all n as a function of a and L.

Solution: 2cm

$$y[n] = \begin{cases} 0 & n < 0 \\ \frac{1-a^{n+1}}{1-a} & 0 \leq n < L \\ a^{n-L+1}\frac{1-a^L}{1-a} & L \leq n \end{cases}$$

3. (24 points) Suppose that the frequency response of a system is:

$$H(\omega) = \begin{cases} 67 & -\frac{\pi}{5} \le \omega \le \frac{2\pi}{5} \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $h[n]$, the impulse response of this system, for all n.

Solution: 2cm All of the following are acceptable solutions:

$$\begin{aligned} h[n] &= \frac{67}{2\pi jn}\left(e^{jn2\pi/5} - e^{-jn\pi/5}\right) \\ &= \frac{67}{\pi n} e^{jn\pi/10}\sin(3\pi n/10) \\ &= 67 \times \frac{3}{10} e^{jn\pi/10}\text{sinc}(3\pi n/10) \end{aligned}$$

4. (24 points) Consider the following signal:

$$w[n] = \begin{cases} 0.1 & -27 \leq n < -12 \\ 0 & \text{otherwise} \end{cases}$$

Note that the DTFT of this signal can be expressed as $W(\omega) = e^{j\omega b} R(w)$ for some real-valued scalar b and real-valued function $R(\omega)$. Find b and $R(\omega)$.

Solution: 2cm

$$b = 20$$

$$R(\omega) = 0.1 \frac{\sin(15\omega/2)}{\sin(\omega/2)}$$

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