UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal Processing
EXAM 1

October 11, 2024

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not leave any unresolved integrals or summations in your answer. Other than that, however, you should not simplify explicit numerical expressions.
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ____

NetID: ____

Phasors

$$A\cos(2\pi ft+\theta)=\Re\left\{Ae^{j\theta}e^{j2\pi ft}\right\}=\frac{1}{2}e^{-j\theta}e^{-j2\pi ft}+\frac{1}{2}e^{j\theta}e^{j2\pi ft}$$

Spectrum

$$\textbf{Scaling:}\;\; y(t)=Gx(t)=\sum_{k=-N}^{N}(Ga_k)\,e^{j2\pi f_k t}$$

$$\textbf{Add a Constant:}\;\; y(t)=x(t)+C=(a_0+C)+\sum_{k\neq 0}a_k e^{j2\pi f_k t}$$

$$\textbf{Add Signals:}\;\; \text{If } f_k=f'_n=f''_m \;\text{ then }\; a_k=a'_n+a''_m$$

$$\textbf{Time Shift:}\;\; y(t)=x(t-\tau)=\sum_{k=-N}^{N}\left(a_k e^{-j2\pi f_k\tau}\right)e^{j2\pi f_k t}$$

$$\textbf{Frequency Shift:}\;\; y(t)=x(t)e^{j2\pi Ft}=\sum_{k=-N}^{N}a_k e^{j2\pi(f_k+F)t}$$

$$\textbf{Differentiation:}\;\; y(t)=\frac{dx}{dt}=\sum_{k=-N}^{N}(j2\pi f_k a_k)\,e^{j2\pi f_k t}$$

Fourier Series

$$\textbf{Analysis:}\;\; X_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-j2\pi kt/T_0}dt$$

$$\textbf{Synthesis:}\;\; x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j2\pi kt/T_0}$$

Sampling and Interpolation:

$$x[n]=x\left(t=\frac{n}{F_s}\right)$$

$$f_a=\min\left(f \bmod F_s,-f \bmod F_s\right)$$

$$z_a=\begin{cases} z & f \bmod F_s<-f \bmod F_s\\ z^* & f \bmod F_s>-f \bmod F_s\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}y[n]p(t-nT_s)$$

1. (25 points) Suppose that

$$x(t) = 2\cos(2\pi 400t) + 5\sin(2\pi 1000t)$$
$$y(t) = 5\cos(2\pi 1000t) + 2\sin(2\pi 2000t)$$
$$z(t) = x(t) + y(t)$$

What is the spectrum of $z(t)$? Recall that a spectrum is defined to be a list of frequencies with their corresponding phasors, including both positive and negative frequencies.

Solution:

$$z(t) = 2\cos(2\pi 400t) + M\cos(2\pi 1000t + \theta) + 2\cos\left(2\pi 2000t - \frac{\pi}{2}\right),$$

where $Me^{j\theta} = 5 + 5e^{-j\frac{\pi}{2}} = 5 - 5j = 5\sqrt{2}e^{-j\frac{\pi}{4}}$, so

$$\text{frequencies} = \{-2000, -1000, -400, 400, 1000, 2000\}$$
$$\text{phasors} = \left\{e^{j\frac{\pi}{2}}, \frac{5\sqrt{2}}{2}e^{j\frac{\pi}{4}}, 1, 1, \frac{5\sqrt{2}}{2}e^{-j\frac{\pi}{4}}, e^{-j\frac{\pi}{2}}\right\}$$

2. (25 points) Consider the following periodic signal:

$$x(t) = \begin{cases} 5e^{-400t} & 0 \le t < 0.01 \\ x(t-0.01) & \text{otherwise} \end{cases}$$

Find the Fourier series coefficients of this signal, X_k, including X_0. The only variable appearing in your answer should be k, but your answer may also include numerical constants such as e, π, and j.

Solution: For $k = 0$, we have

$$\begin{aligned} X_0 &= \frac{1}{0.01}\int_0^{0.01} x(t)dt \\ &= 100\int_0^{0.01} 5e^{-400t}dt \\ &= \frac{1}{-4}\left[5e^{-400t}\right]_0^{0.01} \\ &= -\frac{5}{4}\left(e^{-4}-1\right) \end{aligned}$$

For other values of k, we have

$$\begin{aligned} X_k &= \frac{1}{0.01}\int_0^{0.01} x(t)e^{-j2\pi 100kt}dt \\ &= 100\int_0^{0.01} 5e^{-(400+j2\pi 100k)t}dt \\ &= -\frac{100}{400+j2\pi 100k}\left[5e^{-(400+j2\pi 100k)t}\right]_0^{0.01} \\ &= -\frac{5}{4+j2\pi k}\left(e^{-(4+j2\pi k)}-1\right) \end{aligned}$$

3. (25 points) Consider the signal

$$x_a(t) = \cos\left(2\pi 3300t + \frac{\pi}{6}\right) + \cos\left(2\pi 7300t + \frac{\pi}{4}\right) + \cos\left(2\pi 8300t + \frac{\pi}{3}\right)$$

Suppose that this signal is sampled at 8000 samples/second, to create the signal $x[n]$:

$$x[n] = \cos(\omega_1 n + \theta_1) + \cos(\omega_2 n + \theta_2) + \cos(\omega_3 n + \theta_3)$$

Find values of ω_k and θ_k, $k \in \{1, 2, 3\}$, such that $0 \leq \omega_k \leq \pi$ and $x[n] = x_a(n/8000)$.

Solution:

$$\begin{aligned} x[n] &= \cos\left(2\pi\frac{3300n}{8000} + \frac{\pi}{6}\right) + \cos\left(2\pi\frac{7300n}{8000} + \frac{\pi}{4}\right) + \cos\left(2\pi\frac{8300n}{8000} + \frac{\pi}{3}\right) \\ &= \cos\left(2\pi\frac{3300n}{8000} + \frac{\pi}{6}\right) + \cos\left(2\pi\frac{700n}{8000} - \frac{\pi}{4}\right) + \cos\left(2\pi\frac{300n}{8000} + \frac{\pi}{3}\right), \end{aligned}$$

so the frequencies and phases of the sampled sinusoids are:

$$\begin{aligned} \omega_1, \omega_2, \omega_3 &= \frac{2\pi 3300}{8000}, \frac{2\pi 700}{8000}, \frac{2\pi 300}{8000} \\ \theta_1, \theta_2, \theta_3 &= \frac{\pi}{6}, -\frac{\pi}{4}, \frac{\pi}{3} \end{aligned}$$

4. (25 points) Consider the following discrete-time signal, $x[n]$:

$$x[n] = \sin\left(\frac{2\pi n}{8}\right),$$

thus $x[n]$ is periodic with a period of 8, and its first 8 samples are $\left\{0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}\right\}$. Suppose that $x_a(t) = \sum_{n=-\infty}^{\infty} x[n]p(t-nT)$, where $T = 0.001$ seconds, and $p(t)$ is a triangular interpolation kernel:

$$p(t) = \max\left(1 - 1000|t|, 0\right)$$

(a) (15 points) Sketch the period of $x_a(t)$ for $0 \le t \le 0.008$, or else specify its values using an equation.

Solution:

$$x_a(t) = \begin{cases} 1000t\frac{\sqrt{2}}{2} & 0 \le t \le 0.001 \\ 1000\left(1 - \frac{\sqrt{2}}{2}\right)(t - 0.001) + \frac{\sqrt{2}}{2} & 0.001 \le t \le 0.002 \\ 1000\left(\frac{\sqrt{2}}{2} - 1\right)(t - 0.002) + 1 & 0.002 \le t \le 0.003 \\ 1000\left(-\frac{\sqrt{2}}{2}\right)(t - 0.003) + \frac{\sqrt{2}}{2} & 0.003 \le t \le 0.005 \\ 1000\left(-1 + \frac{\sqrt{2}}{2}\right)(t - 0.005) - \frac{\sqrt{2}}{2} & 0.005 \le t \le 0.006 \\ 1000\left(-\frac{\sqrt{2}}{2} + 1\right)(t - 0.006) - 1 & 0.006 \le t \le 0.007 \\ 1000\left(\frac{\sqrt{2}}{2}\right)(t - 0.007) - \frac{\sqrt{2}}{2} & 0.007 \le t \le 0.008 \end{cases}$$

(b) (10 points) What frequencies are present in the spectrum of $x_a(t)$?

Solution: Its spectrum includes the frequencies $1000k \pm 125$Hz, for every integer value of k.

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