UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 401 Signal and Image Analyais
Spring 2023

EXAM 2

Monday, October 30, 2023

- This is a CLOSED BOOK exam.
- You are permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- Do not simplify explicit numerical expressions. The expression "$e^{-5}\cos(3)$" is a MUCH better answer than "-0.00667".
- If you're taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 1:00pm; you will need to photograph and upload your answers by exactly 2:00pm.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Name: ______________________________

Convolution

$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Frequency Response

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$h[n] * \cos(\omega n) = |H(\omega)| \cos\left(\omega n + \angle H(\omega)\right)$$

Rectangular Window and Ideal LPF

$$w_R[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = e^{-\frac{j\omega(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

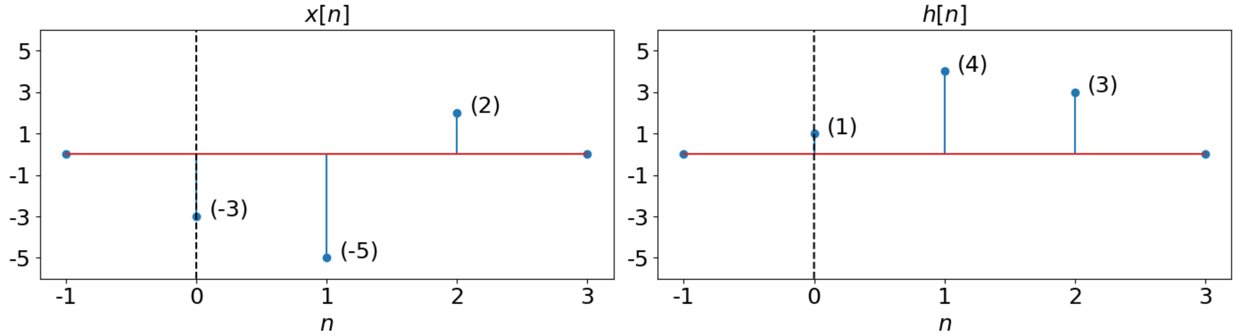
$$h_{\text{ideal}}[n] = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n) \leftrightarrow H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Hamming Window

$$w_H[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

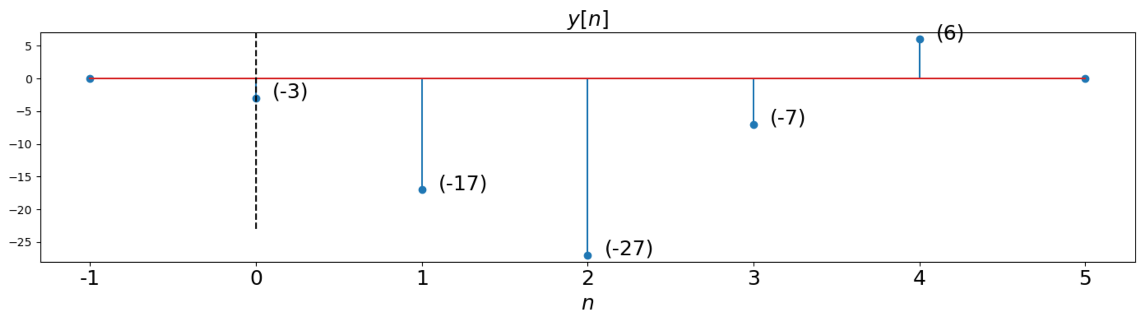
$$W_H(\omega) = 0.54 W_R(\omega) + 0.23 W_R\left(\omega - \frac{2\pi}{N-1}\right) + 0.23 W_R\left(\omega + \frac{2\pi}{N-1}\right)$$

1. (19 points) The following figure shows two signals, $x[n]$ and $h[n]$. Each signal has three nonzero samples, with values as shown in the figure.



Find $y[n] = h[n] * x[n]$. Plot your answer using a stem plot similar to the plot shown above, and label the value of each nonzero sample in parentheses next to the sample, as shown above.

Solution:



2. (32 points) Frequency modulation (FM) is a system that accepts an input signal $x[n]$, and computes an output signal $y[n]$ according to

$$y[n] = e^{j(\omega_0 + x[n])n},$$

where ω_0 is a real scalar constant called the carrier frequency.

(a) Is FM linear? Prove your answer.

Solution:

$$x_1[n] \to y_1[n] = e^{j(\omega_0 + x_1[n])n}$$
$$x_2[n] \to y_2[n] = e^{j(\omega_0 + x_2[n])n}$$
$$y_1[n] + y_2[n] = e^{j(\omega_0 + x_1[n])n} + e^{j(\omega_0 + x_2[n])n}$$
$$x_3[n] = x_1[n] + x_2[n] \to y_3[n] = e^{j(\omega_0 + x_1[n] + x_2[n])n}$$
$$y_3[n] \neq y_1[n] + y_2[n]$$

Therefore FM is not a linear system.

(b) Is FM shift-invariant? Prove your answer.

Solution:

$$x_1[n] \to y_1[n] = e^{j(\omega_0 + x_1[n])n}$$
$$y_1[n - n_0] = e^{j(\omega_0 + x_1[n-n_0])(n-n_0)}$$
$$x_2[n] = x_1[n - n_0] \to y_2[n] = e^{j(\omega_0 + x_1[n-n_0])n}$$
$$y_2[n] \neq y_1[n - n_0]$$

Therefore FM is not a shift-invariant system.

(c) Is FM causal? Prove your answer.

Solution: The output, $y[n] = e^{j(\omega_0 + x[n])n}$, depends only on the value of $x[n]$ at the same time. It does not depend on any future values of $x[n]$, therefore the system is causal.

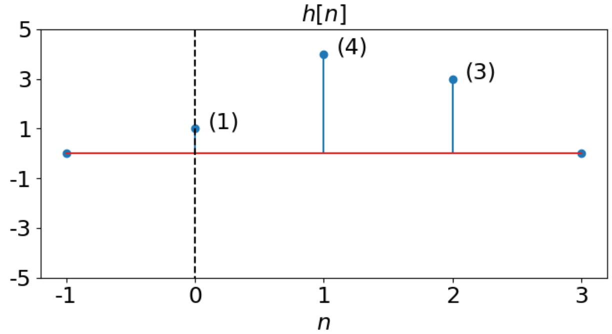
(d) Is FM stable? Prove your answer.

Solution: For any finite-valued input, $|x[n]| < \infty$,

$$|y[n]| = |e^{j(\omega_0 + x[n])n}| = 1$$

In other words, bounded inputs always yield bounded outputs, so the system is stable.

3. (19 points) Suppose the signal $x[n] = e^{j\omega_0 n}$ is passed through a linear shift-invariant system with the following impulse response:



Find the system output. Your answer should be a function of only n and ω_0.

Solution:

$$y[n] = \left(1 + 4e^{-j\omega_0} + 3e^{-j2\omega_0}\right) e^{j\omega_0 n},$$

4. (30 points) This problem is about the design of a lowpass filter.

(a) The desired frequency response is

$$F(\omega) = \begin{cases} 1 & |\omega| \leq \frac{\pi}{5} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

What is $f[n]$?

Solution:

$$f[n] = \left(\frac{1}{5}\right) \text{sinc}\left(\frac{\pi n}{5}\right)$$

(b) As a first step toward implementation, the filter is truncated, thus

$$g[n] = \begin{cases} f[n] & |n| \leq 64 \\ 0 & \text{otherwise} \end{cases}$$

The resulting frequency response can be writtten as

$$G(\omega) = \frac{1}{2\pi} F(\omega) * W(\omega), \tag{2}$$

where $F(\omega)$ is the function shown in Eq. (1), and $W(\omega)$ is some function of ω. Find $W(\omega)$ as a function of ω.

Solution:

$$W(\omega) = \frac{\sin(129\omega/2)}{\sin(\omega/2)}$$

(c) As a second step toward implementation, the filter is delayed by 64 samples, thus

$$h[n] = g[n - 64]$$

The resulting frequency response can be written as $H(\omega) = D(\omega)G(\omega)$, where $G(\omega)$ is the function shown in Eq. (2), and $D(\omega)$ is some function of ω. Find $D(\omega)$ as a function of ω.

Solution:

$$D(\omega) = e^{-j64\omega}$$

THIS PAGE IS SCRATCH PAPER.